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Long-Range $\mathfrak{psu}(2,2|4)$ Bethe Ansätze for Gauge Theory and Strings

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In Honor of Hans Bethe

Abstract

We generalize various existing higher-loop Bethe ansätze for simple sectors of the integrable long-range dynamic spin chain describing planar $\mathcal{N}=4$ Super Yang-Mills Theory to the full $\mathfrak{psu}(2,2|4)$ symmetry and, asymptotically, to arbitrary loop order. We perform a large number of tests of our conjectured equations, such as internal consistency, comparison to direct three-loop diagonalization and expected thermodynamic behavior. In the special case of the $\mathfrak{su}(1|2)$ subsector, corresponding to a long-range t-J model, we are able to derive, up to three loops, the S-matrix and the associated nested Bethe ansatz from the gauge theory dilatation operator. We conjecture novel all-order S-matrices for the $\mathfrak{su}(1|2)$ and $\mathfrak{su}(1,1|2)$ subsectors, and show that they satisfy the Yang-Baxter equation. Throughout the paper, we muse about the idea that quantum string theory on $AdS_5 \times S^5$ is also described by a $\mathfrak{psu}(2,2|4)$ spin chain. We propose asymptotic all-order Bethe equations for this putative "string chain", which differ in a systematic fashion from the gauge theory equations.

1 Introduction and Overview

Recently a powerful new tool for the study of planar non-abelian gauge theories and strings on curved space-times, as well as the conjectured dualities linking the two, has become available. *Integrability* has made its appearance in $\mathcal{N} = 4$ Super Yang-Mills theory and in IIB string theory on the $AdS_5 \times S^5$ background. It is beginning to shed entirely new light on the AdS/CFT duality. Proving or disproving part of the gauge/string correspondence suddenly seems to be within reach. The central new tool is a technique widely known as the *Bethe ansatz*. It dates back to the year 1931 when Hans Bethe solved the Heisenberg spin chain in his pioneering work [1]. Its impact on condensed matter theory and mathematical physics cannot be underestimated.

The first, crucial observation in the context of the gauge/string duality was made by Minahan and Zarembo [2]. They noticed that the conformal quantum operators in the scalar field sector of $\mathcal{N} = 4$ gauge theory are, at the planar one-loop level, in one-to-one correspondence with the translationally invariant eigenstates of an integrable $\mathfrak{so}(6)$ magnetic quantum spin chain. The spin chain Hamiltonian corresponds to the gauge theoretic planar one-loop dilatation operator, whose "energy" eigenvalues yield the scaling weights of the conformal operators. This observation turned out to be a first hint at a very deep structure. The result generalizes to all local operators of the planar one-loop $\mathcal{N} = 4$ theory [3]. What is more, evidence was found that integrability extends beyond the one-loop approximation [4].

First indications that planar gauge theories may contain hidden integrable structures were discovered in a QCD context in seminal work by Lipatov [5]. References to further interesting work on integrability in QCD may be found in [6]. New aspects of the more recent developments [2–4] when comparing to these important earlier insights are that (*i*) the integrability links space-time to internal symmetries, (*ii*) the studied spin chains allow for an interesting thermodynamics with a large number of lattice sites, (*iii*) the integrability extends beyond the leading approximation and leads to novel long-range spin chains, (*iv*) it allows for comparison with similar structures appearing in the conjectured dual string theory.

And indeed it was argued in [7] that superstrings on $AdS_5 \times S^5$ are classically integrable. This allows, in many cases, to find explicit solutions of the non-linear equations describing the classical motions of strings on that background [8]. More generally, classical integrability even permits to describe generic classical motions of the string as solutions of algebraic curves [9–14]. A question of primary importance is clearly whether this classical integrability extends to the quantum theory. First encouraging evidence was presented in [15] where a Bethe ansatz for the string sigma model was found "experimentally" in a special case. Excitingly, this Bethe ansatz also stems from a long-range spin chain, as first noticed in [16].

We may therefore hope to gain a much deeper understanding of the AdS/CFT correspondence by directly comparing the integrable structures of gauge and string theory as opposed to considering only the spectrum of energies. This is the approach proposed and pursued for semiclassical strings in [17, 9–14] and for quantum strings in [15, 16, 18]. It leads to a deeper probe of earlier proposals for comparing the string and gauge theory in the plane wave/BMN [19] and the semiclassical limit [20] (see [21] and [22] for qualitative

and quantitative precursors in particular cases). Reviews of the work on these proposals are found in [23] and [24, 25]. The immense usefulness of the Bethe ansatz in the study of these proposals was demonstrated in [2] and [26].

In the case of gauge theory one may directly demonstrate the emergence of an integrable long-range spin chain from the first few orders of perturbation theory [2–4, 27, 28]. Of course, with current technology it is not possible to give an all-orders, let alone non-perturbative, proof. However, constructing the correct chain under some reasonable assumptions does not appear to be entirely out of reach; for first steps in this direction see [29]. In the case of string theory the evidence for an underlying spin chain structure is entirely indirect [15, 16, 18]. If true, it should emerge from an exact quantization of the string sigma model. Despite some progress towards setting up the quantization of the integrable model [30] it is fair to say that this putative "string chain" is currently hiding well.

In this paper we will continue the construction of higher-loop Bethe ansätze, begun in [28, 29, 15, 16, 18], for the integrable long-range spin chains which appear to describe gauge and (possibly) string theory. The approach we shall follow is somewhat similar to the one commonly applied for the solution of a jigsaw puzzle. We attempt to self-consistently assemble smaller building blocks (sectors) into an emergent larger picture, until we end up with a proposal for the full set of asymptotic Bethe equations. An important restriction is the condition of asymptoticity: We suspect that our equations diagonalize the underlying spin chains only to $\mathcal{O}(g^{2L})$, where g is the coupling constant and L the chain length. While our derivation contains multiple gaps in need of proof, we feel that we have performed many of the currently possible consistency checks without finding any manifest contradictions. This includes checks against direct diagonalization of the three-loop Hamiltonian (where available), thermodynamic consistency, symmetry, and the idea [15, 18], supported by the structure found in [27], that the asymptotic S-matrices of string and gauge theory differ by a global, flavor-independent "dressing factor":

$$S^{\mathrm{string}} = \hat{S}^{\mathrm{dressing}}\, S^{\mathrm{gauge}}\,. \tag{1.1}$$

Hopefully this factor will appear as one goes from weak (gauge theory) to strong (string theory) coupling and will reconcile the notorious third-order discrepancies noticed in [31, 28] between string and gauge theory in, respectively, the near-BMN and Frolov-Tseytlin limits. For an interesting alternative proposal to explain the discrepancy, see [32].

We start out in Section 2 by reconsidering the S-matrices and associated Bethe equations of the three two-component sectors $\mathfrak{su}(2)$, $\mathfrak{su}(1|1)$ and $\mathfrak{sl}(2)$. We first review their Hamiltonians and the perturbative asymptotic Bethe ansatz (PABA) of [18] which is based, to a large extent, on the Bethe's original work [1]. In the case of gauge theory we extend the three-loop asymptotic S-matrices of [18] for $\mathfrak{su}(1|1)$ and $\mathfrak{sl}(2)$ to all loops, in analogy with the $\mathfrak{su}(2)$ case [29]. On the string side we improve the approximate S-matrices of [18] to (hopefully) asymptotically exact ones. In particular, the improved Bethe equations appear to be consistent with the existence of an integrable spin chain for quantum strings beyond the $\mathfrak{su}(2)$ sector [15, 16]. Our construction confirms, for string and gauge theory, the following relation between the S-matrices of the three sectors [18]:

$$S_{\mathfrak{sl}(2)} = S_{\mathfrak{su}(1|1)}\, S^{-1}_{\mathfrak{su}(2)}\, S_{\mathfrak{su}(1|1)}\,. \tag{1.2}$$

Note that (1.1) is consistent with our claim that (1.2) holds for both string and gauge theory.

In Section 3 we consider the unification of the $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$ sectors into the $\mathfrak{su}(1|2)$ sector.[1] The one-loop Hamiltonian of the latter is identical to the one of an integrable quantum super spin chain important in condensed matter theory: the so-called t-J model. We then extract, using the PABA, the three-loop S-matrix from the Hamiltonian of [27]. Using the insights from Section 2 we generalize this S-matrix to all loops (asymptotically). Excitingly, the resulting S-matrix (3.26), which we have not been able to find in the literature, satisfies the Yang-Baxter equation. We end this section by applying the machinery of the nested Bethe ansatz to our S-matrix, thereby deriving two equivalent sets of nested long-range $\mathfrak{su}(1|2)$ Bethe equations.

In Section 4 we consider the further unification of all three two-component sectors of Section 2 into a super spin chain with non-compact symmetry $\mathfrak{su}(1,1|2)$. Alternatively we can say that we combine the $\mathfrak{su}(1|2)$ sector of the previous section with the derivative sector $\mathfrak{sl}(2) = \mathfrak{su}(1,1)$. Here, as in [18], the higher loop Hamiltonian, and thus the PABA, is currently not available. Upon inspection of the one-loop S-matrix we however find a very natural all-loop generalization in complete analogy with the $\mathfrak{su}(1|2)$ case. We then apply the nested Bethe ansatz and derive the system of Bethe equations for this sector. There are four forms of the Bethe equations corresponding to different Cartan matrices for $\mathfrak{su}(1,1|2)$ which we prove to be equivalent by dualization as in [14]. Incidentally this yields infinitely many novel higher-loop predictions for gauge theory operators, some of which should be testable, at least at the two-loop level, against field theory computations.

In the final Section 5 we take a big leap and attempt to extend the previous system of $\mathfrak{su}(1,1|2)$ Bethe equations to the full set of excitations with symmetry $\mathfrak{psu}(2,2|4)$. Here we have not yet found an appropriate S-matrix; this is not only due to increased complexity, but mainly related to the fact that the higher-loop $\mathcal{N}=4$ spin chain is *dynamic* [27]. The nested Bethe ansatz in this case will most likely require some new ideas. There are two main sources of inspiration which lead to our equations: The finite gap solution of the string sigma model [12] gives expressions which we can generalize beyond the thermodynamic limit using our experience from the two-component sectors. The possibility of dualizing the Bethe equations between several equivalent Cartan matrices seems to imply a very special structure of the equations. Our final set of equations is displayed in Tab. 5. It agrees with a list of properties outlined in [25]. Most importantly, we find an intriguing symmetry of the higher-loop equations which appears to be related to the dynamic nature of the underlying chain.

[1] Conventionally, we will always place the number referring to spacetime symmetries before the number referring to internal symmetry.

2 Rank-One Sectors

In $\mathcal{N}=4$ SYM there are three sectors of local operators where the dilatation operator takes a particularly simple form. They are all based on a vacuum state

$$|0,L\rangle = \mathcal{Z}^L \tag{2.1}$$

which is half-BPS and therefore has exactly vanishing anomalous dimension. Here $\mathcal{Z}$ is a complex combination of two real scalars of the theory. The excitations of the vacuum are obtained by changing some of the $\mathcal{Z}$'s into other fields. In the $\mathfrak{su}(2)$ sector we replace $\mathcal{Z}$ by another complex scalar $\mathcal{X}$. The $\mathfrak{su}(1|1)$ sector has fermionic excitations $\mathcal{U}$. In the third $\mathfrak{sl}(2)=\mathfrak{su}(1,1)$ sector the excitations are covariant derivatives $\mathcal{DZ}$. Here, unlike the other two cases, it is allowed to have multiple excitations ($\mathcal{D}^n\mathcal{Z}$) residing at a single site.

2.1 Review of Hamiltonians

The symmetry algebra of conformal $\mathcal{N}=4$ SYM is $\mathfrak{psu}(2,2|4)$, it acts linearly on the set of states. This representation $\mathfrak{J}(g)$ depends on the coupling constant g defined by

$$g^2 = \frac{g_{\mathrm{YM}}^2 N}{8\pi^2} = \frac{\lambda}{8\pi^2}\,. \tag{2.2}$$

In this paper we consider a small coupling constant g and apply perturbation theory. Classically, i.e. at $g=0$, the action of the symmetry algebra on the set of states is merely the tensor product of the action on the individual sites. This is how the symmetry algebra acts for common quantum spin chains. When we turn on interactions, the picture changes: The range of the action of $\mathfrak{J}(g)$ extends; for each loop order, i.e. order in $\lambda \sim g^2$, the generators may act on one additional neighboring site at the same time. Moreover, they may even create or destroy spin chain sites and one might therefore consider the spin chain as *dynamic* [27].

A priori, there is no natural Hamiltonian $\mathcal{H}$ for the spin chain, there is only the symmetry algebra. However, in perturbation theory one can identify the anomalous dimension $\delta\mathfrak{D}(g)$ as a $\mathfrak{u}(1)$ generator which commutes with $\mathfrak{psu}(2,2|4)$; we shall set

$$\delta\mathfrak{D}(g) = g^2\mathcal{H}(g), \tag{2.3}$$

because by definition the classical part of the anomalous dimension vanishes and for various reasons we would like to have finite energies E at $g=0$. These are consequently related to the anomalous dimension by

$$\delta D(g) = g^2 E(g). \tag{2.4}$$

In all investigated cases it has turned out that in the planar limit there are additional generators $\mathcal{Q}_r$ of a form similar to the one of $\mathcal{H}$. All of them commute with each other, with $\mathfrak{psu}(2,2|4)$ and with the Hamiltonian. This is an implication of the apparent higher-loop integrability of $\mathcal{N}=4$ SYM [4]. In fact, the first two charges $\mathcal{Q}_1, \mathcal{Q}_2$ are crucial for

L	$(E_0, E_2, E_{4\mathrm{g}} \vert E_{4\mathrm{s}})^P$
3	$(6, -12, 42 \vert 33)^+$
4	$(4, -6, 17 \vert 13)^-$
5	$(10E - 20, -17E + 60, \frac{117}{2}E - 230 \vert \frac{107}{2}E - 210)^+$
6	$(6, -\frac{21}{2}, \frac{555}{16} \vert \frac{483}{16})^-$
	$(2, -\frac{3}{2}, \frac{37}{16} \vert \frac{29}{16})^-$
7	$(14E^2 - 56E + 56, -23E^2 + 172E - 224, 79E^2 - 695E + 966 \vert 74E^2 - 653E + 910)^+$
8	$(4, -5, \frac{49}{4} \vert \frac{41}{4})^-$
	$(8E - 8, -13E + 18, \frac{179}{4}E - 61 \vert \frac{167}{4}E - 57)^-$

Table 1: Spectrum of lowest-lying two-excitation states.

L	K	$(E_0, E_2, E_{4\mathrm{g}} \vert E_{4\mathrm{s}})^P$
$5+1$	3	$(6, -9, \frac{63}{2} \vert \frac{63}{2})^-$
$6+1$	3	$(5, -\frac{15}{2}, 25 \vert \frac{95}{4})^\pm$
$7+1$	3	$(4, -5, 14 \vert 13)^\pm$
		$(6, -9, 33 \vert 33)^-$
	4	$\big(20E^2 - 116E + 200, -32E^2 + 340E - 800, 112E^2 - 1400E + 3600 \big\vert 101E^2 - 1304E + 3400\big)^+$
$8+1$	3	$\big(17E^2 - 90E + 147, -\frac{51}{2}E^2 + \frac{525}{2}E - \frac{1239}{2}, \frac{169}{2}E^2 - \frac{2091}{2}E + \frac{5649}{2} \big\vert 82E^2 - \frac{2037}{2}E + \frac{11025}{4}\big)^\pm$
	4	$(5, -\frac{15}{2}, \frac{55}{2} \vert \frac{109}{4})^\pm$
		$(12E - 24, -18E + 54, 57E - 171 \vert 48E - 147)^-$

Table 2: Spectrum of lowest-lying states of the $\mathfrak{su}(2)$ sector.

extracting physical information from the theory. The second charge is nothing but the Hamiltonian and the eigenvalues therefore match

$$\mathcal{H} = \mathcal{Q}_2, \qquad E = Q_2. \tag{2.5}$$

The first charge is the logarithm of the spin chain shift operator. The shift operator permutes the spin chain state cyclically by one step.[2] As gauge theory states are identified cyclically, the shift operator must act trivially on all physical states

$$\exp(i\mathcal{Q}_1) \simeq 1, \qquad Q_1 = 2\pi m. \tag{2.6}$$

In fact this condition is essential for the consistency of the model [25]. Finally, the higher charges do not seem to contain interesting physical information. They serve as hidden symmetries leading to integrability.

2.2 Spectrum

Our aim is to determine the spectrum of the Hamiltonian and thus the spectrum of anomalous dimensions. The energies of the states can be obtained directly from the Hamiltonian. In practice, this requires that the state is not too complicated. More

[2] The shift operator is graded and generates the appropriate signs for commuting fermions.

L	K	$(E_0, E_2, E_{4\mathrm{g}} \vert E_{4\mathrm{s}})^P$
5	4	$(10, -20, \frac{145}{2} \vert \frac{125}{2})^-$
6	3	$(8, -14, 49 \vert 47)^\pm$
	4	$(8, -14, 46 \vert 38)^+$
7	3	$(7, -12, \frac{83}{2} \vert \frac{153}{4})^\pm$
	4	$\big(28E^2-252E+728, -51E^2+906E-3864, 179E^2-3965E+20090 \big\vert 160E^2-3629E+18634\big)^-$
	6	$(14, -28, \frac{203}{2} \vert \frac{175}{2})^+$
8	3	$(6, -\frac{19}{2}, \frac{247}{8} \vert \frac{223}{8})^\pm$
		$(8, -\frac{29}{2}, \frac{427}{8} \vert \frac{419}{8})^\pm$
	4	$(8, -14, \frac{97}{2} \vert \frac{89}{2})^+$
		$(10, -18, 64 \vert 61)^\pm$
		$(16E-56, -26E+170, \frac{165}{2}E-638 \vert \frac{139}{2}E-554)^+$
	5	$(12, -22, 77 \vert 69)^\pm$
	6	$(12, -22, 75 \vert 63)^-$

Table 3: Spectrum of lowest-lying states of the $\mathfrak{su}(1|1)$ sector.

importantly, we have to get hold of the Hamiltonian in the first place. At the one-loop level the complete Hamiltonian was obtained in [33]. The largest piece of the higher-loop Hamiltonian is known for the $\mathfrak{su}(2|3)$ sector from [27]. We will obtain all our spectral data from this particular Hamiltonian and display it in a number of tables, e.g. Tab. 1. A set of states is specified by the length L, the number of excitations K and the leading three orders of the energy

$$E_{\mathrm{g}} = E_0 + g^2 E_2 + g^4 E_{4\mathrm{g}} + \ldots, \qquad E_{\mathrm{s}} = E_0 + g^2 E_2 + g^4 E_{4\mathrm{s}} + \ldots \tag{2.7}$$

as well as the (charge conjugation/spin chain inversion) parity P. We shall distinguish between two models, gauge theory, with energy E_{g}, and a string chain, with energy E_{s}, see below. The parity may be either $+$ or $-$. Many states, however, come in exactly degenerate pairs, but with opposite parity [4]. These pairs are indicated by $\pm$ and are a direct consequence of the existence of a conserved, parity-inverting charge $\mathcal{Q}_3$. When there is only a single state with given L and K, the state cannot mix and the energy always expands in rational numbers. When mixing with other states occurs, however, the energies are often irrational.[3] In those cases we prefer to encode the energies as the roots of an algebraic equation. For mixing of M states, the tables state, in the form $(X_0(E), X_2(E), X_{4\mathrm{g}}(E)|X_{4\mathrm{s}}(E))$, a polynomial $X(E) = X_0(E) + X_2(E)g^2 + X_4(E)g^4$ of degree $M-1$. The energies of the M states are obtained as solutions to

$$E^M = X(E). \tag{2.8}$$

If desired, numerical values may be obtained immediately with an appropriate root finder. E.g. with `Mathematica` one might use, to an accuracy of k digits,

[3] Note however that the energies cannot be transcendental, as one might have expected from general experience with perturbative quantum field theory. The reason is that the higher-loop Bethe ansatz always leads to algebraic equations for the energies.

```
Series[e /. NSolve[e^M == X[e], e, k], {g,0,4}] // Normal // Chop.
```

In this section we concentrate on subsectors of the full theory where the main part of the symmetry algebra has rank one. These are the $\mathfrak{su}(2)$, $\mathfrak{su}(1|1)$ and $\mathfrak{sl}(2)$ sectors introduced above. In later sections we will extend the analysis to larger sectors and eventually to the complete model. The $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$ sectors are contained in the $\mathfrak{su}(2|3)$ sector of [27] for which we know the three-loop Hamiltonian. We display the energies of the lowest-lying spin chain states in Tab. 1,2,3. Some of these results have appeared in [4, 27] and we have supplemented the values of $E_{4\mathrm{s}}$. Note that the two sectors intersect on the set of states with only two excitations [34], we display those states separately in Tab. 1. For the $\mathfrak{sl}(2)$ sector a higher-loop Hamiltonian is currently not available.

2.3 Review of Two-Component Bethe Ansätze

In this section we will review the perturbative asymptotic Bethe ansatz (PABA) developed in [18]. This is a Bethe ansatz for spin chain excitations in position space which is very closely related to the original ansatz by Bethe [1], but adapted to the long-range Hamiltonians introduced in Sec. 2.1. It starts with the assumption that cyclic chains of finite length L are infinite chains with periodic boundary conditions. We shall therefore consider an infinitely long chain and try to determine its eigenstates. Only later we will restrict to periodic states which are given by the solutions to the Bethe equations.

The vacuum state is a tensor product of fields $\mathcal{Z}$ much like the ferromagnetic vacuum of a magnetic chain

$$|0\rangle = |\ldots\mathcal{Z}\mathcal{Z}\mathcal{Z}\ldots\rangle. \tag{2.9}$$

It is a protected state, its energy is exactly zero

$$\mathcal{H}|0\rangle = 0. \tag{2.10}$$

We may place a few excitations $\mathcal{A} = \mathcal{X},\mathcal{U},\mathcal{D}\mathcal{Z}$ (depending on the sector) into the vacuum and then try to find the eigenstates of the Hamiltonian. Let us start with a single excitation

$$|\ldots\mathcal{Z}\overset{\overset{\ell}{\downarrow}}{\mathcal{A}}\mathcal{Z}\ldots\rangle = \alpha^\dagger_\ell|0\rangle. \tag{2.11}$$

The latter representation indicates that we can view the excitation as being produced by some creation operator $\alpha^\dagger_\ell$ acting on site ℓ. The Hamiltonian is homogeneous and the appropriate ansatz for an eigenstate is a plane wave with momentum p

$$|p\rangle = \sum_\ell e^{ip\ell}\,\alpha^\dagger_\ell|0\rangle. \tag{2.12}$$

It is automatically an eigenstate $\mathcal{H}|p\rangle = e(p)|p\rangle$ and the energy turns out to be [28]

$$e(p) = 4\sin^2(\tfrac{1}{2}p) - 8g^2\sin^4(\tfrac{1}{2}p) + 32g^4\sin^6(\tfrac{1}{2}p) + \ldots \tag{2.13}$$

which is consistent with the all-loop prediction [29, 19]

$$e(p) = g^{-2}\sqrt{1 + 8g^2 \sin^2(\tfrac{1}{2}p)} - g^{-2}. \tag{2.14}$$

Now we can attack the two-excitation problem, the eigenstates are given by[4]

$$|p_1, p_2\rangle = \sum_{\ell_1,\ell_2} \Psi_{\ell_1,\ell_2}(p_1, p_2)\, \alpha^\dagger_{\ell_1} \alpha^\dagger_{\ell_2} |0\rangle. \tag{2.15}$$

with some wave function $\Psi_{\ell_1,\ell_2}(p_1, p_2)$. At each fixed loop order the range of the interaction is finite. Asymptotically, the wave function should therefore factorize into one-particle wave functions with

$$\Psi_{\ell_1,\ell_2}(p_1, p_2) = e^{ip_1\ell_1 + ip_2\ell_2}\, A \qquad \text{for} \qquad \ell_1 \ll \ell_2 \tag{2.16}$$

and

$$\Psi_{\ell_1,\ell_2}(p_1, p_2) = e^{ip_1\ell_1 + ip_2\ell_2}\, A' \qquad \text{for} \qquad \ell_1 \gg \ell_2 \tag{2.17}$$

where A and A' are independent of ℓ_1, ℓ_2. The elements of the wave function in the interaction range $\ell_1 \approx \ell_2$ are determined by the non-diffractive scattering problem

$$\mathcal{H}|p_1, p_2\rangle = \big(e(p_1) + e(p_2)\big)|p_1, p_2\rangle. \tag{2.18}$$

It also fixes the ratio of A and A'. The wave function is unphysical,[5] the relevant physical information is the phase shift Φ between the wave function on both sides of the interaction

$$S(p_2, p_1) = \exp\big(i\Phi(p_2, p_1)\big) = \frac{A'}{A}\,. \tag{2.19}$$

The amplitude $S(p_2, p_1)$ is the two-body *S-matrix*.

For an integrable Hamiltonian $\mathcal{H}$ the phase shift $\Phi(p_1, p_2)$ is all we need to know to construct the asymptotic state with arbitrarily many excitations. Here asymptotic means that we neglect those contributions to the exact wave function where some excitations are sufficiently close to each other to interact. These contributions are determined from the asymptotic data by the Hamiltonian, but they are not relevant for finding the spectrum of energies. An eigenstate is specified by a set of K momenta p_k

$$|\{p_k\}\rangle = \sum_{\ell_k} A_{\ell_k} \prod_{k=1}^{K} \Big(\exp(ip_k\ell_k)\, \alpha^\dagger_{\ell_k}\Big)\, |0\rangle. \tag{2.20}$$

Here the amplitudes A_{ℓ_k} depend asymptotically only on the ordering $\sigma(\ell_k)$ of the (well-separated) positions ℓ_k of the excitations. The amplitudes are related among each other

[4]The excitation generators $\alpha^\dagger$ can be either bosonic or fermionic. According to their grading, they will automatically generate some relative signs in the states $|\ldots\mathcal{ZAZ}\ldots\mathcal{ZAZ}\ldots\rangle$.

[5]In any renormalization scheme there is freedom of applying a linear transformation on the set of all local operators; this is part of how divergencies are absorbed in a quantum field theory. Physical information must not depend on the change of basis. For example, anomalous dimensions and correlation functions of *eigen*operators are invariant quantities while the wave function depends on the basis.

through the phase shifts (up to one overall constant): If two orderings σ and σ' are related by the interchange of two adjacent excitations, the amplitudes A_σ and $A_{\sigma'}$ must be related by (2.19). Then the energy of the (corresponding exact) eigenstate is given by

$$E = \sum_{k=1}^{K} e(p_k). \tag{2.21}$$

If the chain were really infinitely long this would already be the end of the analysis, and arbitrary values of the momenta p_k would yield valid solutions. For a chain of length L we however need to take into account the periodicity conditions. In particular, if we shift the position of any particular excitation by L lattice sites the wave function (2.20) should not change. These K constraints lead to a set of K *Bethe equations*:

$$\exp(iLp_k) = \prod_{\substack{j=1\\ j\neq k}}^{K} \exp\bigl(i\mathit{\Phi}(p_k, p_j)\bigr) = \prod_{\substack{j=1\\ j\neq k}}^{K} S(p_k, p_j)\,. \tag{2.22}$$

Finally, we have to take into account that the eigenvalue of the lattice shift operator should equal 1 for the translationally invariant states we are interested in. This leads, from (2.6), to the momentum constraint

$$\prod_{k=1}^{K} \exp(ip_k) = 1 \qquad \text{i.e.} \qquad Q_1 = \sum_{k=1}^{K} p_k = 2\pi m. \tag{2.23}$$

This completes our general review of the Bethe ansatz technique for two-component systems. The case of more than two components is conceptually similar, but significantly more involved and will be discussed in the following chapters.

Let us now return to the concrete case of the two-component sectors relevant to $\mathcal{N} = 4$ gauge theory. For the $\mathfrak{su}(2)$ sector, the all-loop scattering phase was conjectured in [29] to be given by

$$\exp\bigl(i\mathit{\Phi}_{\mathcal{X}}(p_k, p_j)\bigr) = \frac{u(p_k) - u(p_j) + i}{u(p_k) - u(p_j) - i} \tag{2.24}$$

with the rapidity function

$$u(p) = \tfrac{1}{2}\cot(\tfrac{1}{2}p)\sqrt{1 + 8g^2\sin^2(\tfrac{1}{2}p)}\,. \tag{2.25}$$

This may be proven to three loops by embedding the $\mathfrak{su}(2)$ Hamiltonian to $\mathcal{O}(g^4)$ into the Inozemtsev spin chain [28]. For this long-range system exact wave functions are known, and the correct phase shift may therefore be extracted. At four and five loops the conjectured gauge theory Hamiltonian[6] may no longer be embedded into the Inozemtsev model.

[6] The conjecture is based on assuming integrability, proper scaling in the thermodynamic limit and certain features of field-theoretic perturbation theory [29,25]. No rigorous proof for BMN scaling exists beyond three loops. This means that, strictly speaking, the Inozemtsev model has not yet been completely ruled out, even though we strongly suspect that the Hamiltonian of [29,25] is indeed the correct one. The conjectured phase (2.24) may be verified from the latter to five-loop order by the PABA of [18] (M. S., T. Klose, unpublished).

The scattering phase for the fermionic $\mathfrak{su}(1|1)$ sector was distilled from the $\mathfrak{su}(2|3)$ vertex of [27] by the PABA ("perturbative asymptotic Bethe ansatz") technique in [18]. It reads

$$\begin{aligned}\varPhi_{\mathcal{U}}(p_k, p_j) &= 4g^2 \sin(\tfrac{1}{2}p_k)\sin(\tfrac{1}{2}p_j)\sin(\tfrac{1}{2}p_k - \tfrac{1}{2}p_j)\\ &\quad + g^4 \sin(\tfrac{1}{2}p_k)\sin(\tfrac{1}{2}p_j)\Big(-7\sin(\tfrac{1}{2}p_k - \tfrac{1}{2}p_j) + \sin(\tfrac{1}{2}p_k - \tfrac{3}{2}p_j)\\ &\quad\quad + \sin(\tfrac{3}{2}p_k - \tfrac{1}{2}p_j) + \sin(\tfrac{3}{2}p_k - \tfrac{3}{2}p_j)\Big) + \mathcal{O}(g^6).\end{aligned} \tag{2.26}$$

In the derivative $\mathfrak{sl}(2)$ sector the PABA is currently not applicable since we are lacking the Hamiltonian beyond the one-loop level. However, in [18] the simple relation (1.2) between the S-matrices was discovered from a spectroscopic analysis of strings in the near-plane wave background. It was then assumed that (1.2) should also hold in gauge theory. In view of (2.19) this led to the following conjecture for the scattering phase for $\mathfrak{sl}(2)$

$$\varPhi_{\mathcal{D}}(p_k, p_j) = 2\varPhi_{\mathcal{U}}(p_k, p_j) - \varPhi_{\mathcal{X}}(p_k, p_j). \tag{2.27}$$

This turned out to be consistent with the anomalous dimensions of twist-two operators which are rigorously known to two loops [35] and were conjectured, based on a fully-fledged QCD loop calculation [36], to three loops in [37].[7] An involved two-loop test, using sophisticated superspace Feynman rules (see also [38]), for the simplest twist-three operator was recently successfully performed in [39].

2.4 The Spectral Parameter Plane

In [29] an alternative parametrization of the $\mathfrak{su}(2)$ Bethe ansatz was presented which simplified many expressions. It is based on the map between the rapidity (u) plane and a spectral parameter (x) plane[8]

$$x(u) = \tfrac{1}{2}u + \tfrac{1}{2}u\sqrt{1 - 2g^2/u^2}\,, \qquad u(x) = x + \frac{g^2}{2x}\,. \tag{2.28}$$

The relation to the momentum (p) plane is given by

$$\exp(ip) = \frac{x(u + \frac{i}{2})}{x(u - \frac{i}{2})}\,. \tag{2.29}$$

Let us for simplicity define several equivalent parametrizations of Bethe roots. We shall consider the spectral parameter x_k as fundamental. The momentum p_k, rapidity u_k and shifted spectral parameters x_k^+ and x_k^- are defined as

$$u_k = u(x_k), \qquad x_k^{\pm} = x(u_k \pm \tfrac{i}{2}), \qquad p_k = -i\log\frac{x_k^+}{x_k^-}. \tag{2.30}$$

[7] Here, the asymptotic S-matrix (2.27) works even better than expected [18]: For two and three loops, the chain is already *shorter* than the range of the interaction, but the Bethe ansatz still reproduces the correct result. This curiosity is explained by the results of Sec. 4.5 which relate the $L = 2$ spin chain to a $L = 4$ spin chain by supersymmetry. Then the interaction is just sufficiently short up to three loops.

[8] The map $x(u)$ has two branches. For $g \approx 0$ we will pick the branch where $x \approx u$. The other branch is given by $x' = g^2/2x$.

The local charges Q_r of the integrable model can now be conveniently expressed as

$$Q_r = \sum_{k=1}^{K} q_r(x_k), \qquad q_r(x_k) = \frac{i}{r-1}\left(\frac{1}{(x_k^+)^{r-1}} - \frac{1}{(x_k^-)^{r-1}}\right), \tag{2.31}$$

with the regularized first charge $q_1(x_k) = -i\log(x_k^+/x_k^-) = p_k$ being the momentum. Particularly important are the first two charges, the total momentum Q_1 for the momentum constraint and the energy Q_2 for the anomalous dimension δD

$$Q_1 = 2\pi m, \qquad \delta D = g^2 Q_2. \tag{2.32}$$

Before we discuss the Bethe equations let us note some useful identities relating the u- and x-plane

$$\begin{aligned} u_k - u_j &= (x_k - x_j)(1 - g^2/2x_k x_j) \\ &= (x_k^\pm - x_j^\pm)(1 - g^2/2x_k^\pm x_j^\pm), \\ u_k - u_j \pm \tfrac{i}{2} &= (x_k^\pm - x_j)(1 - g^2/2x_k^\pm x_j) \\ &= (x_k - x_j^\mp)(1 - g^2/2x_k x_j^\mp), \\ u_k - u_j \pm i &= (x_k^\pm - x_j^\mp)(1 - g^2/2x_k^\pm x_j^\mp). \end{aligned} \tag{2.33}$$

They are easily confirmed using the definition of $u(x)$ in (2.28).

2.5 Bethe Equations for Spins

Let us next attempt to express the Bethe ansätze of Sec. 2.3 through the spectral parameters x^+, x^-. We shall discover that this allows to find, in analogy with the $\mathfrak{su}(2)$ case [29], very natural all-loop extensions of the three-loop S-matrices for the $\mathfrak{su}(1|1)$ and $\mathfrak{sl}(2)$ sectors.

The Bethe equations (2.22,2.24) for the $\mathfrak{su}(2)$ sector in the u-plane read

$$\left(\frac{x(u_k + \frac{i}{2})}{x(u_k - \frac{i}{2})}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{K} \frac{u_k - u_j + i}{u_k - u_j - i}\,. \tag{2.34}$$

Using the identities (2.33) we can translate them to the $x^\pm$-plane

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{K} \frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\,\frac{1 - g^2/2x_k^+ x_j^-}{1 - g^2/2x_k^- x_j^+}\,. \tag{2.35}$$

As it stands this result is neither remarkable nor very helpful. It turns out, however, that the second term agrees *precisely* with the function $\exp(i\mathit{\Phi}_\mathcal{U})$ of (2.26)

$$\exp\bigl(i\mathit{\Phi}_\mathcal{U}(x_k, x_j)\bigr) = \frac{1 - g^2/2x_k^+ x_j^-}{1 - g^2/2x_k^- x_j^+} \tag{2.36}$$

at third loop order $\mathcal{O}(g^4)$ up to which the function $\Phi_{\mathcal{U}}$ is known from [18, 27]. This form thus appears to be a natural asymptotic generalization of $\Phi_{\mathcal{U}}$. For simplicity of notation, we shall *assume* that the $\mathfrak{su}(1|1)$ sector of gauge theory at higher loops is indeed described by this scattering phase. We do not have as much justification for this point of view as for the $\mathfrak{su}(2)$ sector, where some calculations up to $\mathcal{O}(g^{10})$ exist [25], but below we shall see that it neatly fulfills some non-trivial requirements. Therefore, the asymptotic generalization of the Bethe equations (2.22,2.26) for the $\mathfrak{su}(1|1)$ sector of gauge theory apparently reads

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{K} \frac{1-g^2/2x_k^+x_j^-}{1-g^2/2x_k^-x_j^+}\,. \tag{2.37}$$

Finally, assuming again (1.2), we find that the asymptotic[9] form of the conjectured Bethe equation (2.22,2.27) for the $\mathfrak{sl}(2)$ sector should be given by

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{K} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\,\frac{1-g^2/2x_k^+x_j^-}{1-g^2/2x_k^-x_j^+}\,. \tag{2.38}$$

We can combine the asymptotic Bethe equations for all three sectors in the concise form

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{K} \left(\frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\right)^{\eta}\,\frac{1-g^2/2x_k^+x_j^-}{1-g^2/2x_k^-x_j^+}\,. \tag{2.39}$$

Here the parameter η specifies the sector: $\eta=+1$ for $\mathfrak{su}(2)$, $\eta=0$ for $\mathfrak{su}(1|1)$ or $\eta=-1$ for $\mathfrak{sl}(2)$. For all three sectors, gauge theory states obey the momentum constraint

$$\prod_{k=1}^{K} \frac{x_k^+}{x_k^-} = 1 \tag{2.40}$$

and their anomalous dimension is given by

$$\delta D = g^2 \sum_{k=1}^{K} \left(\frac{i}{x_k^+} - \frac{i}{x_k^-}\right)\,. \tag{2.41}$$

As discussed in [29, 18], the three-loop spectrum obtained from (2.39) for $\eta = 1, 0$ agrees with a large number of states obtained by direct diagonalization of the Hamiltonian, cf. Tab. 1,2,3.

[9] Curiously the Bethe ansatz (2.38) works even better than one might have expected [18] as noted above. While maybe not too likely, it is not excluded that (2.38) reproduces the anomalous dimensions of twist-two operators to all orders in perturbation theory! We hope that an appropriate four-loop field theory computation will be performed in the future.

2.6 Bethe Equations for Strings

The quantization of IIB string theory in the curved $AdS_5 \times S^5$ geometry is currently not understood. The AdS/CFT conjecture proposes that free strings moving on that background should correspond to *planar* $\mathcal{N} = 4$ gauge theory. More precisely, it holds that the energies and eigenstates of free strings should map to, respectively, scaling dimensions and eigenstates of the gauge theory planar dilatation operator. In turn, the latter appear to be given by the energies and eigenstates of certain novel long-range quantum spin chains. Assuming the, at least approximate, validity of the correspondence, and assuming that the spin chains do not spontaneously evaporate as one goes from weak to strong coupling, we conclude that *quantum strings on* $AdS_5 \times S^5$ *should also be described by an integrable long-range spin chain.*

For the best studied case of the $\mathfrak{su}(2)$ sector, Bethe equations for this "string chain" were proposed in [15]. These were based on a discretization of the finite gap equation describing the classical sigma model in this sector [9]. Shortly thereafter it was demonstrated that these equations indeed diagonalize, to at least five orders in the coupling constant, a long-range $\mathfrak{su}(2)$ spin chain similar to the one describing weakly coupled gauge theory [16]. Using the variables introduced above, the equations read

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{j=1\\j\neq k}}^{K} \frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\,\frac{1 - g^2/2x_k^+ x_j^-}{1 - g^2/2x_k^- x_j^+}\,\sigma^2(x_k, x_j) \tag{2.42}$$

with the "stringy" scattering term [15]

$$\sigma(x_k, x_j) = \exp\big(i\theta(x_k, x_j)\big), \tag{2.43}$$

where the phase is given by

$$\theta(x_k, x_j) = \sum_{r=2}^{\infty} \big(\theta_{r,r+1}(x_k, x_j) - \theta_{r+1,r}(x_k, x_j)\big), \tag{2.44}$$

with

$$\theta_{r,s}(x_k, x_j) = (\tfrac{1}{2}g^2)^{(r+s-1)/2}\, q_r(x_k)\, q_s(x_j). \tag{2.45}$$

This term may also be summed and expressed through the spectral parameters as [16]

$$\sigma(x_k, x_j) = \frac{1 - \dfrac{g^2}{2x_k^- x_j^+}}{1 - \dfrac{g^2}{2x_k^+ x_j^-}} \left(\frac{1 - \dfrac{g^2}{2x_k^- x_j^+}}{1 - \dfrac{g^2}{2x_k^+ x_j^+}}\, \frac{1 - \dfrac{g^2}{2x_k^+ x_j^-}}{1 - \dfrac{g^2}{2x_k^- x_j^-}} \right)^{i(u_k - u_j)}. \tag{2.46}$$

The stringy scattering term modifies the spectrum of the gauge theory spin chain at three-loop order $\mathcal{O}(g^6)$. The lowest-lying three-loop energies of the two similar but distinct spin chains are found in Tab. 1,2, and may be obtained either by direct diagonalization or by solving the Bethe equations (2.42), where $\sigma(x_k, x_j) = 1$ for gauge theory and $\sigma(x_k, x_j)$ as in (2.46) for the string chain.

What about the remaining two-component sectors $\mathfrak{su}(1|1)$ and $\mathfrak{sl}(2)$? In [18] the approach of [15] was extended to these cases. An approximate stringy S-matrix was obtained for $\mathfrak{sl}(2)$ from a discretization of the finite gap equation describing the classical sigma model in this sector [10]. Furthermore, it was argued that the spectrum of strings in the near-plane wave geometry [40], which had recently be obtained in [31, 41], is consistent with a factorized S-matrix. The latter was extracted for the three two-component sectors, yielded a stringy S-matrix for $\mathfrak{su}(1|1)$, and agreed with the $\mathfrak{sl}(2)$ discretization. A comparison of the three obtained S-matrices then led to the relation (1.2).

However, the obtained S-matrices were only designed, by construction, to reproduce the string results for the near-BMN and Frolov-Tseytlin limits, i.e. the $\mathcal{O}(1/L)$ terms in the S-matrix. In contradistinction to the $\mathfrak{su}(2)$ case, the $\mathfrak{su}(1|1)$ and $\mathfrak{sl}(2)$ S-matrices lacked periodicity in the momenta and could therefore not exactly correspond to a quantum spin chain. The existence of such a spin chain encompassing all sectors was nevertheless conjectured, along with the proposal (1.1), which says that the full S-matrix of the string and the gauge chain should differ by an overall multiplicative, flavor-independent dressing factor. From the above $\mathfrak{su}(2)$ results we then find the factor to be

$$\hat{S}^{\text{dressing}}(x_k, x_j) = \sigma^2(x_k, x_j)\,, \tag{2.47}$$

with $\sigma(x_k, x_j)$ as in (2.46). Given its rather complicated structure we are not sure whether we have already found the final, analytically exact form of this dressing factor. In fact, one would hope that this is not the case; we would prefer an interpolating function which smoothly changes from $\sigma = 1$ for the weakly coupled gauge theory[10] to σ as given in (2.46) at strong coupling [15].

Given the conjecture (2.39) in Sec. 2.5 we may then write a Bethe ansatz for the string chain in all three sectors:

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{K} \left(\frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\right)^\eta \frac{1 - g^2/2x_k^+ x_j^-}{1 - g^2/2x_k^- x_j^+}\,\sigma^2(x_k, x_j)\,. \tag{2.48}$$

A very important test of this proposal is that to three loop order (2.48) agrees with direct diagonalization of the $\mathfrak{su}(1|1)$ string chain Hamiltonian as discussed in [18]. The latter is known from [27, 16].

It would be very interesting to compare the $1/L$ corrections to the Frolov-Tseytlin limit [43] on the gauge and string side using our Bethe equations (2.48), cf. [44].

For the remainder of this paper we will always include a dressing factor $\sigma(x_k, x_j)$ in the Bethe equations and assume $\sigma(x_k, x_j) = 1$ for gauge theory or the expression in (2.46) for the string chain.

[10] In fact, we currently cannot exclude a dressing factor for gauge theory which sets in at higher loop orders or non-perturbatively. For example, as in [42], a violation of proper scaling beyond three loops, which has not yet been completely ruled out, might be induced by a non-trivial σ.

2.7 Two-Excitation States

One non-trivial property the Bethe ansätze for all three sectors should satisfy is related to two-excitation states. These states form multiplets of the superconformal algebra [34] which have the unique property of having representatives in all three sectors. For these states, the different Bethe equations should therefore reproduce the same values for the energies (and higher charges). The two Bethe roots are given by $x_1 = x$ and $x_2 = -x$ due to the momentum constraint (2.40). The Bethe equation now reads

$$\left(\frac{x^+}{x^-}\right)^L = \left(\frac{x^+ + x^+}{x^- + x^-}\right)^\eta \frac{1 + g^2/2x^+x^+}{1 + g^2/2x^-x^-}\,\sigma^2(x, -x). \tag{2.49}$$

where we have made use of $x_2^\pm = -x^\mp$. It is now clear that a state of length L in the $\mathfrak{su}(2)$ sector with $\eta = +1$ has a corresponding state of length $L-1$ in the $\mathfrak{su}(1|1)$ sector with $\eta = 0$ and a corresponding state of length $L-2$ in the $\mathfrak{su}(1,1)$ sector with $\eta = -1$. This is true for the Bethe ansätze for spins as well as for strings, in generalization of the results obtained in [18].

2.8 Thermodynamic Limit

Let us consider the thermodynamic limit of very long spin chains $L \to \infty$ with $g = \mathcal{O}(L)$ while keeping the energy E small. Here we distinguish between two cases:

- The near-BMN/plane-wave limit [40, 31] with a fixed number of excitations and $E = \mathcal{O}(1/L^2)$. The spectrum is described by a scattering problem.
- The Frolov-Tseytlin limit [20] with $\mathcal{O}(L)$ excitations and $E = \mathcal{O}(1/L)$. The spectrum is described by the spectral curve or equivalently by a Riemann-Hilbert problem.

Using the expressions in App. A.1 it is straightforward to read off the phases for the pairwise scattering from (2.48). For gauge theory with $\sigma(x_k, x_j) = 1$ we find

$$\varPhi = 2\eta\varPsi + (1-\eta)(2\theta + \theta_{1,2} - \theta_{2,1}). \tag{2.50}$$

Here $\varPsi$ is the main scattering phase

$$\varPsi(x_k, x_j) = -i\log\frac{u_k - u_j - i/2}{u_k - u_j + i/2} = \frac{1}{u_j - u_k} + \mathcal{O}(1/L^2) \tag{2.51}$$

and the auxiliary phases, cf. (2.44,2.45), are given by

$$\begin{aligned}
\theta(x_k, x_j) &= \frac{g^2/2x_k^2}{1 - g^2/2x_k^2}\,\frac{g^2/2x_j^2}{1 - g^2/2x_j^2}\,\frac{1/x_j - 1/x_k}{1 - g^2/2x_jx_k} + \mathcal{O}(1/L^2),\\
\theta_{1,2}(x_k, x_j) &= \frac{1/x_k}{1 - g^2/2x_k^2}\,\frac{g^2/2x_j^2}{1 - g^2/2x_j^2} + \mathcal{O}(1/L^2),\\
\theta_{2,1}(x_k, x_j) &= \frac{g^2/2x_k^2}{1 - g^2/2x_k^2}\,\frac{1/x_j}{1 - g^2/2x_j^2} + \mathcal{O}(1/L^2).
\end{aligned} \tag{2.52}$$

The corresponding scattering term for the string chain is

$$\begin{aligned}\varPhi &= 2\eta\varPsi - 2\eta\theta + (1-\eta)(\theta_{1,2} - \theta_{2,1}).\\ &= 2\eta\psi + (1+\eta)\theta_{1,2} - (1-\eta)\theta_{2,1}.\end{aligned} \tag{2.53}$$

with ψ the main scattering phase in the x-plane

$$\psi(x_k, x_j) = -i\log\frac{x_k - x_j^+}{x_k - x_j^-} = \frac{1}{1 - g^2/2x_j^2}\,\frac{1}{x_j - x_k} + \mathcal{O}(1/L^2). \tag{2.54}$$

The total phase $\varPhi$ agrees precisely with the asymptotic form for near plane-wave strings derived in [18].

For many, $K = \mathcal{O}(L)$, excitations, the Bethe equations turn into integral equations

$$2\eta \not{H}(x) + F(x) = -2\pi n_a, \qquad \text{for } x \in \mathcal{C}_a. \tag{2.55}$$

The functions $H(x), G(x)$ are two different types of resolvents

$$H(x) = \int_{\mathcal{C}} \frac{dy\,\rho(y)}{u(y) - u(x)}\,, \qquad G(x) = \int_{\mathcal{C}} \frac{dy\,\rho(y)}{1 - g^2/2y^2}\,\frac{1}{y - x}\,, \tag{2.56}$$

and a slash implies a principal value prescription, $\not{H}(x) = \frac{1}{2}H(x+\varepsilon) + \frac{1}{2}H(x-\varepsilon)$. The potential $F(x)$ for gauge theory reads

$$F_{\mathrm{g}}(x) = \frac{L/x}{1 - g^2/2x^2} + (1-\eta)\left(2G(g^2/2x) - \frac{G'(0)\,g^2/2x}{1 - g^2/2x^2} - \frac{G(0)\,(2 - g^2/2x^2)}{1 - g^2/2x^2}\right) \tag{2.57}$$

and for string theory we obtain

$$F_{\mathrm{s}}(x) = \frac{L/x}{1 - g^2/2x^2} - 2\eta G(g^2/2x) + 2\eta G(0) + \frac{(1+\eta)G'(0)g^2/2x}{1 - g^2/2x^2} - \frac{(1-\eta)G(0)g^2/2x^2}{1 - g^2/2x^2}. \tag{2.58}$$

Note that we can write the resulting integral equation for string chains as

$$2\eta\not{G}(x) + \frac{(1+\eta)g^2/2x}{1 - g^2/2x^2}\,G'(0) - \frac{(1-\eta)g^2/2x^2}{1 - g^2/2x^2}\,G(0) + \frac{L/x}{1 - g^2/2x^2} = -2\pi n_a, \quad \text{for } x \in \mathcal{C}_a \tag{2.59}$$

This expression agrees precisely with the classical string sigma model in [9, 10, 12].

The potentials F_{g} and F_{s} are similar for small values of g. This leads to an agreement between gauge theory and string theory up to two loops. Let us try to explain the agreement using general features of the Bethe equations. The expansion of the Bethe equation for Bethe root x around $x = \infty$ usually yields the Noether charges. Here, $2\eta H(x) + F(x)$ yields the single global charge of the symmetry group of the the model in both cases

$$2\eta H(x) + F(x) = \frac{1}{x}\big(L + 2\eta K + \tfrac{1}{2}(1-\eta)\,\delta D\big) + \mathcal{O}(1/x^2). \tag{2.60}$$

When we combine this feature with proper scaling of x we see that an expansion in $1/x$ is equivalent to an expansion in g^2. The agreement of F_{g} and F_{s} up to $\mathcal{O}(1/x)$ leads to an agreement of the Bethe equations at $\mathcal{O}(g^2)$. For the energy shifts it means that gauge theory and string theory agree up to $\mathcal{O}(g^4)$, i.e. two loops. Starting at three loops the Bethe equations differ due to $F_{\mathrm{s}} - F_{\mathrm{g}} \sim 1/x^2 \sim g^4$.

2.9 Strong-Coupling Limit

One of the fascinating features of the $\mathfrak{su}(2)$ Bethe ansatz for quantum strings (2.42) is that it quantitatively reproduces the expected strong coupling behavior of anomalous dimensions [15]. It was found that this behavior is, for two-excitation states with mode number n

$$D = D_0 + g^2 E = 2\sqrt[4]{\lambda n^2}\,. \tag{2.61}$$

For multi-excitation states the formula is identical, with $n = \sum_{k=1}^{K_+} n_k$ where K_+ is the number of modes with positive mode number $n_k > 0$. It turned out that to this leading order in strong coupling, the result was entirely due to the scattering effects resulting from the dressing factor, i.e. the relevant Bethe equation, cf. (2.47,2.46), leading to (2.61) is

$$\prod_{\substack{j=1\\ j\neq k}}^{K} \hat{S}^{\text{dressing}}(x_k, x_j) = \prod_{\substack{j=1\\ j\neq k}}^{K} \sigma^2(x_k, x_j) = 1\,. \tag{2.62}$$

Since our Bethe ansatz for the string chain in the different sectors differs from gauge theory as in (1.1), and since it is easy to see that S^{gauge} is always subleading[11] we see that (2.61) immediately generalizes to those sectors. In fact, getting ahead of ourselves, in light of (1.1) it will turn out to be true for the entire $\mathfrak{psu}(2,2|4)$ string chain, in line with general expectations [45].

3 The Factorized S-Matrix of the $\mathfrak{su}(1|2)$ Sector

In the $\mathcal{N} = 4$ gauge theory the planar $\mathfrak{su}(1|2)$ sector consists of operators of the type

$$\operatorname{Tr} \mathcal{U}^{K_1} \mathcal{X}^{K_2-K_1} \mathcal{Z}^{L-K_2} + \ldots, \tag{3.1}$$

where $\mathcal{Z}$ and $\mathcal{X}$ are two out of the three complex adjoint scalars of the $\mathcal{N} = 4$ model, and $\mathcal{U}$ is an adjoint gaugino (in $\mathcal{N} = 1$ connotation). The dots indicate that we need to consider all possible orderings of the fields inside the trace, and diagonalize the set of such operators with respect to dilatation. As in the previous chapter, this is most easily done when interpreting the dilatation operator as a Hamiltonian acting on a spin chain of length L. This requires opening up the trace and replacing it by a quantum mechanical state on a one-dimensional lattice of L sites:

$$\cancel{\operatorname{Tr}}(\mathcal{Z}\mathcal{Z}\mathcal{X}\mathcal{U}\ldots\mathcal{X}\mathcal{Z}) \longrightarrow |\mathcal{Z}\mathcal{Z}\mathcal{X}\mathcal{U}\ldots\mathcal{X}\mathcal{Z}\rangle. \tag{3.2}$$

The fact that the original operators (3.1) are single-trace has two consequences for the spin chain interpretation, to be distinguished: (i) The trace links the matrix indices of the first and the last constituent field. The boundary conditions of the spin chain are

[11] Strictly speaking this is not true for the scattering among left-movers and the scattering among right-movers, i.e. when $p_k \cdot p_j > 0$. It is however easy to see, cf. [15], that the resulting phase shifts sum to zero upon substitution into the dispersion relation.

therefore periodic. (*ii*) The trace has the property of cyclicity. This means that only the subset of translationally invariant states of the spin chain are relevant to gauge theory.[12]

A new feature compared to Section 2 is that we now have a three-component system. If we choose the fields $\mathcal{Z}$ as our reference (=vacuum) configuration, the fields $\mathcal{X}$, $\mathcal{U}$ may be regarded as two distinct types of excitations of the chain. The total number of excitations is K_2.

As in Section 2 we will assume that a modified version of the spin chain describing weakly coupled gauge theory applies to strongly coupled quantum string theory. This is very natural, as we can think of the $\mathfrak{su}(1|2)$ sector as a "unification" of the $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$ sectors of the previous sectors. The deformation which takes us from gauge to string theory in these (compact) two-component sectors immediately "lifts" to their three-component unification. It is nevertheless important to keep in mind that we do not currently know how to find the deformed spin chain, i.e. the analog of (3.1), directly from the quantum string sigma model.

3.1 One-Loop Scattering in the $\mathfrak{su}(1|2)$ Sector

The planar one-loop Hamiltonian in the closed $\mathfrak{su}(1|2)$ sector reads

$$\mathcal{H}_0 = \sum_{\ell=1}^{L} (1 - \mathit{\Pi}_{\ell,\ell+1}) \,. \tag{3.3}$$

It may be extracted from the complete one-loop $\mathcal{N} = 4$ dilatation operator [33] and rewritten with the help of the graded permutation operator $\mathit{\Pi}_{\ell,\ell+1}$ which exchanges the partons at sites ℓ and $\ell + 1$, picking up a minus sign if the exchange involves two fermions $\mathcal{U}$.

This sector unifies the $\mathfrak{su}(2)$ and the $\mathfrak{su}(1|1)$ subsectors of the previous Section 2. At one loop it again corresponds to an important nearest-neighbor integrable spin chain of condensed matter theory, the famous supersymmetric t-J model believed to be relevant to the understanding of high T_c superconductivity [46]. Its supersymmetry was apparently first noticed in [47]. It was first solved by the coordinate-space Bethe ansatz in [48] and by the algebraic Bethe ansatz in [49]. An equivalent lattice gas model was solved much earlier by a coordinate-space Bethe ansatz [50]. As a warm-up for the long-range case, let us briefly review the latter method, extending the explanations of [18] to the three-component case.

Since the Hamiltonian (3.3) is integrable, it again suffices to consider two-body states in order to derive the many-body S-matrix. The cases of two $\mathcal{X}$'s or two $\mathcal{U}$'s have been considered in [18] and we are left to consider the mixed case of one $\mathcal{X}$ and $\mathcal{U}$ each. These

[12] In the presence of fermions, as in (3.2), we have to use a shift operator which is properly graded.

states are, in an obvious notation

$$
|\Psi\rangle = \begin{pmatrix} |\mathcal{UX}\rangle \\ \\ |\mathcal{XU}\rangle \end{pmatrix} = \begin{pmatrix} \sum_{1\le\ell_1<\ell_2\le L} \Psi_{\mathcal{UX}}(\ell_1,\ell_2)\; |...\mathcal{Z}\overset{\overset{\ell_1}{\downarrow}}{\mathcal{U}}\mathcal{Z}...\mathcal{Z}\overset{\overset{\ell_2}{\downarrow}}{\mathcal{X}}\mathcal{Z}...\rangle \\ \\ \sum_{1\le\ell_1<\ell_2\le L} \Psi_{\mathcal{XU}}(\ell_1,\ell_2)\; |...\mathcal{Z}\overset{\overset{\ell_1}{\downarrow}}{\mathcal{X}}\mathcal{Z}...\mathcal{Z}\overset{\overset{\ell_2}{\downarrow}}{\mathcal{U}}\mathcal{Z}...\rangle \end{pmatrix}, \tag{3.4}
$$

where $\ell_{1,2}$ denotes the positions (with $\ell_1 < \ell_2$) of the two "particles" $\mathcal{U}$ and $\mathcal{X}$ inside the opened trace. In this lattice gas picture the fields $\mathcal{Z}$ are "vacancies", i.e. unoccupied lattice sites. In contrast to the two-component case we now need to take into account that the two excitations are distinguishable. The number of components of the above wave function (3.4) corresponds to their possible orderings (two, for the moment).

Acting with the Hamiltonian (3.3) on the state (3.4) we find the Schrödinger equations $\mathcal{H}_0 \cdot |\Psi_{..}\rangle = E_0\, |\Psi_{..}\rangle$ for the position space wave functions

$$
\begin{aligned}
&\text{for} \quad \ell_2 > \ell_1 + 1 : && (3.5)\\
&\quad E_0\, \Psi_{\mathcal{UX}}(\ell_1,\ell_2) = 2\,\Psi_{\mathcal{UX}}(\ell_1,\ell_2) - \Psi_{\mathcal{UX}}(\ell_1-1,\ell_2) - \Psi_{\mathcal{UX}}(\ell_1+1,\ell_2) + \\
&\qquad\qquad + 2\,\Psi_{\mathcal{UX}}(\ell_1,\ell_2) - \Psi_{\mathcal{UX}}(\ell_1,\ell_2-1) - \Psi_{\mathcal{UX}}(\ell_1,\ell_2+1)\,, \\
&\text{for} \quad \ell_2 = \ell_1 + 1 : && (3.6)\\
&\quad E_0\, \Psi_{\mathcal{UX}}(\ell_1,\ell_2) = 3\,\Psi_{\mathcal{UX}}(\ell_1,\ell_2) - \Psi_{\mathcal{UX}}(\ell_1-1,\ell_2) - \Psi_{\mathcal{UX}}(\ell_1,\ell_2+1) - \\
&\qquad\qquad - \Psi_{\mathcal{XU}}(\ell_1,\ell_2)\,,
\end{aligned}
$$

with a second set of equations for the wave functions $\Psi_{\mathcal{XU}}(\ell_1,\ell_2)$ which are identical to (3.5,3.6) after exchanging $\mathcal{U} \leftrightarrow \mathcal{X}$ everywhere.

These difference equations are solved by an appropriate Bethe ansatz:

$$
\begin{aligned}
\Psi_{\mathcal{UX}}(\ell_1,\ell_2) &= A_{\mathcal{UX}}\, e^{ip_1\ell_1+ip_2\ell_2} + A'_{\mathcal{UX}}\, e^{ip_2\ell_1+ip_1\ell_2} \\
\Psi_{\mathcal{XU}}(\ell_1,\ell_2) &= A_{\mathcal{XU}}\, e^{ip_1\ell_1+ip_2\ell_2} + A'_{\mathcal{XU}}\, e^{ip_2\ell_1+ip_1\ell_2}\,.
\end{aligned} \tag{3.7}
$$

The idea is that the partons, coming in as an arbitrary mixed state with initial amplitudes $A_{\mathcal{UX}}$, $A_{\mathcal{XU}}$ propagate freely down the trace with fixed momenta p_1,p_2 until they scatter at $\ell_2 = \ell_1 + 1$. Before they do, i.e. when $\ell_2 > \ell_1 + 1$, the Schrödinger equation (3.5) is satisfied for arbitrary amplitudes as long as the dispersion law

$$
E_0 = \sum_{k=1}^{K_2} 4\, \sin^2(\tfrac{1}{2}p_k) \tag{3.8}
$$

(with $K_2 = 2$, for the moment) holds. However, when the particles hit each other they may exchange momenta, and in addition, unlike in the two-component case, exchange their flavors. This scattering process is *non-diffractive* if the individual momenta p_k are separately conserved. The Bethe ansatz (3.7) assumes this to be true in that the outgoing configuration, with amplitudes $A'_{\mathcal{UX}}$, $A'_{\mathcal{XU}}$, is simply added to the wave function. This is clearly still consistent with the generic case $\ell_2 > \ell_1 + 1$ if the energy expression (3.8)

holds. But now we must check whether the amplitudes may be adjusted such that the Schrödinger equation (3.6) at $\ell_2 = \ell_1 + 1$ is satisfied as well. This may be achieved by the ansatz

$$\begin{pmatrix} A'_{\mathcal{X}\mathcal{U}} \\ \\ A'_{\mathcal{U}\mathcal{X}} \end{pmatrix} = \begin{pmatrix} T^{\mathcal{U}\mathcal{X}}_{\mathcal{U}\mathcal{X}}(p_2, p_1) & R^{\mathcal{U}\mathcal{X}}_{\mathcal{X}\mathcal{U}}(p_2, p_1) \\ \\ R^{\mathcal{X}\mathcal{U}}_{\mathcal{U}\mathcal{X}}(p_2, p_1) & T^{\mathcal{X}\mathcal{U}}_{\mathcal{X}\mathcal{U}}(p_2, p_1) \end{pmatrix} \begin{pmatrix} A_{\mathcal{U}\mathcal{X}} \\ \\ A_{\mathcal{X}\mathcal{U}} \end{pmatrix} . \tag{3.9}$$

and one finds, after substituting the Bethe ansatz (3.7) into (3.6)

$$\begin{aligned} T^{\mathcal{U}\mathcal{X}}_{\mathcal{U}\mathcal{X}}(p_1, p_2) = T^{\mathcal{X}\mathcal{U}}_{\mathcal{X}\mathcal{U}}(p_1, p_2) &= \frac{e^{ip_1} - e^{ip_2}}{e^{ip_1+ip_2} - 2e^{ip_2} + 1} , \\ R^{\mathcal{U}\mathcal{X}}_{\mathcal{X}\mathcal{U}}(p_1, p_2) = R^{\mathcal{X}\mathcal{U}}_{\mathcal{U}\mathcal{X}}(p_1, p_2) &= -\frac{(1 - e^{ip_1})(1 - e^{ip_2})}{e^{ip_1+ip_2} - 2e^{ip_2} + 1} . \end{aligned} \tag{3.10}$$

Here T denotes the transmission amplitudes (the particles pass through each other) and R the reflection amplitudes (the particles backscatter). Note that the order of the particles in the outgoing amplitude A' on the l.h.s. of (3.9) is reversed: Indeed, by our conventions, if the particles transmit their order will change. Together with the well-known expressions for the two-body one-loop S-matrices for $\mathcal{X}$-$\mathcal{X}$ scattering and $\mathcal{U}$-$\mathcal{U}$ scattering in the $\mathcal{Z}$ vacuum (see [18] for a derivation in the same elementary fashion as just presented)[13]

$$S^{\mathcal{X}\mathcal{X}}_{\mathcal{X}\mathcal{X}}(p_1, p_2) = -\frac{e^{ip_1+ip_2} - 2e^{ip_1} + 1}{e^{ip_1+ip_2} - 2e^{ip_2} + 1} , \qquad S^{\mathcal{U}\mathcal{U}}_{\mathcal{U}\mathcal{U}}(p_1, p_2) = 1 , \tag{3.11}$$

we now have the full two-body S-matrix. We will write it down in the basis $|\mathcal{X}\mathcal{X}\rangle$, $|\mathcal{U}\mathcal{X}\rangle$, $|\mathcal{X}\mathcal{U}\rangle$, $|\mathcal{U}\mathcal{U}\rangle$, using a vector notation $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$ for these two-body states. We can think of these as the possible configurations of a very short "auxiliary" two-component spin chain. The two-body S-matrix is then an operator acting on this short spin chain and we thus find, in the so-called transmission-diagonal convention,

$$S_{k,j}(p_k, p_j) = \begin{pmatrix} S^{\mathcal{X}\mathcal{X}}_{\mathcal{X}\mathcal{X}}(p_k, p_j) & & & \\ & T^{\mathcal{U}\mathcal{X}}_{\mathcal{U}\mathcal{X}}(p_k, p_j) & R^{\mathcal{U}\mathcal{X}}_{\mathcal{X}\mathcal{U}}(p_k, p_j) & \\ & R^{\mathcal{X}\mathcal{U}}_{\mathcal{U}\mathcal{X}}(p_k, p_j) & T^{\mathcal{X}\mathcal{U}}_{\mathcal{X}\mathcal{U}}(p_k, p_j) & \\ & & & S^{\mathcal{U}\mathcal{U}}_{\mathcal{U}\mathcal{U}}(p_k, p_j) \end{pmatrix} , \tag{3.12}$$

where matrix elements which are zero were left empty.

Integrability now means that, when we consider an arbitrary number of particles $\mathcal{X}$,$\mathcal{U}$, the many-body S-matrix factorizes into a product of the two-body S-matrices (3.12). A necessary consistency condition for this to be true is the Yang-Baxter equation

$$S_{3,2}S_{3,1}S_{2,1} = S_{2,1}S_{3,1}S_{3,2} , \tag{3.13}$$

[13] As opposed to [18] we use the convention that the wave function has implicit factors of -1 from the crossing of fermionic fields. Therefore the element of the S-matrix $S^{\mathcal{U}\mathcal{U}}_{\mathcal{U}\mathcal{U}} = +1$ merely contributes the scattering phase on top of statistics (none here).

along with unitarity $S_{2,1}S_{1,2}=1$, where we have abbreviated $S_{k,j}=S_{k,j}(p_k,p_j)$. These properties may be checked explicitly for our case (3.12) (best done with a symbolic manipulation program) by acting with both sides of (3.13) on the eight-dimensional vector space spanned by three particles of two possible flavors, i.e. the basis of this state space is $|\mathcal{XXX}\rangle$, $|\mathcal{UXX}\rangle$, $|\mathcal{XUX}\rangle$, $|\mathcal{XXU}\rangle$, $|\mathcal{UUX}\rangle$, $|\mathcal{UXU}\rangle$, $|\mathcal{XUU}\rangle$, $|\mathcal{UUU}\rangle$. Again, it is useful to visualize this state space as the one of a short two-component spin chain, this time of length three. An important consequence is that (3.13) allows to extend the Bethe ansatz (3.7) to an arbitrary number of particles. One may check that that the following Bethe wave function is an eigenfunction of the Hamiltonian with eigenvalue (3.8)

$$\Psi_{\ldots\mathcal{XXUX}\ldots}(\ell_1,\ldots,\ell_{K_2})=\sum_\sigma A^{(\sigma)}_{\ldots\mathcal{XXUX}\ldots}\exp\left(i\sum_{k=1}^{K_2}p_{\sigma(k)}\ell_k\right)\,, \tag{3.14}$$

where $\ell_1<\ell_2<\ldots<\ell_{K_2}$. We now need to distinguish all possible orderings of the K_2-K_1 particles of type $\mathcal{X}$ and the K_1 particles of type $\mathcal{U}$. As before we may identify the various configurations with the states $|\ldots\,\mathcal{XXUX}\,\ldots\rangle$ of a shorter (length=K_2) spin chain. The sum in (3.14) runs over all $K_2!$ permutations σ, caused by the scattering, of the particle momenta p_k.

We found the S-matrix by studying the scattering problem on an infinite lattice. The Schrödinger equation is satisfied for any values of the K_2 momenta. They become quantized upon imposing periodic boundary conditions on Bethe's wave function (3.14). If we push any one, say the k'th, of the K_2 particles once around the circle of the chain (i.e. transforming $\ell_k\to\ell_k+L$ in (3.14)) it acquires a free particle phase factor $\exp(ip_kL)$, which gets shifted by the collision with the K_2-1 fellow particles. This procedure leads to the set of K_2 Bethe equations

$$e^{ip_kL}\,|\Psi\rangle=S_{k,k+1}\ldots S_{k,K_2}S_{k,1}\ldots S_{k,k-1}\cdot|\Psi\rangle\,, \tag{3.15}$$

where we have again abbreviated $S_{k,j}=S_{k,j}(p_k,p_j)$. However, note that, unlike in the two-component case, these are *matrix* Bethe equations. We still need to find the states $|\Psi\rangle$ such that (3.15) is simultaneously satisfied for *all* $k=1,\ldots,K_2$, i.e. the eigenvector $|\Psi\rangle$ should *not* depend on k. We may consider the vector $|\Psi\rangle$ as a state in an *inhomogeneous* nearest-neighbor spin chain made from two components. The inhomogeneity is due to the fact that the particles in the short chain carry labels, namely their momenta in the long chain. The (somewhat tedious) diagonalization of this "smaller" spin chain hidden in the original chain will be postponed to Sec. 3.4. The reason is that we shall find that the all-loop case is not a bit harder than the one-loop case!

For simplicity, we change variables from momenta p_k to rapidities $u_k=\frac{1}{2}\cot(\frac{1}{2}p_k)$. The S-matrix (3.12) then becomes, using (3.10),

$$S_{k,j}(u_k,u_j)=\frac{1}{u_k-u_j-i}\begin{pmatrix} u_k-u_j+i & & & \\ & u_k-u_j & i & \\ & i & u_k-u_j & \\ & & & u_k-u_j-i\end{pmatrix}\,. \tag{3.16}$$

Given the insights and findings of Section 2 we could go on and try to immediately write down, using e.g. the rules (2.33), a conjecture for the all-loop (asymptotic) generalization of the one-loop S-matrix (3.16). Focusing on the numerator $u_k - u_j$ of the transmission amplitudes, we however face the problem that there are three combinations which reduce at weak coupling to this expression: $x_k^+ - x_j^+$, $x_k^- - x_j^-$, as well as $x_k - x_j$. Likewise, it is not clear how to "deform" the off-diagonal reflection factors i. Let us therefore go back to the three-loop $\mathfrak{su}(1|2)$ dilatation operator [27] and apply the perturbative asymptotic Bethe ansatz [18] in order to find the proper higher-loop generalization of the S-matrix (3.16).

3.2 Three-Loop S-Matrix Extraction for $\mathfrak{su}(1|2)$

The $\mathfrak{su}(1|2)$ sector is manifestly embedded into the $\mathfrak{su}(2|3)$ sector studied in [27]. If we act with the three-loop Hamiltonian derived in [27] on the Bethe wave function (3.7) we find that it is still a solution if the excitations are "sufficiently far" apart, i.e. iff $\ell_2 > \ell_1 + 3$. It is therefore reasonable to expect that *asymptotically* we still have[14]

$$\begin{aligned}
\Psi_{\mathcal{U}\mathcal{X}}(\ell_1,\ell_2) &\sim A_{\mathcal{U}\mathcal{X}}\, e^{ip_1\ell_1+ip_2\ell_2} + A'_{\mathcal{U}\mathcal{X}}\, e^{ip_2\ell_1+ip_1\ell_2}\,,\\
\Psi_{\mathcal{X}\mathcal{U}}(\ell_1,\ell_2) &\sim A_{\mathcal{X}\mathcal{U}}\, e^{ip_1\ell_1+ip_2\ell_2} + A'_{\mathcal{X}\mathcal{U}}\, e^{ip_2\ell_1+ip_1\ell_2}\,.
\end{aligned} \tag{3.17}$$

The idea to extract the S-matrix of a long-range system from the asymptotic wave function is apparently due to Sutherland [51]. Accordingly, we expect to still be able to infer, using (3.9), the correct transmission and reflection coefficients from (3.17). However, in order to find the asymptotic phase shifts we need to multiply the various exponentials in (3.17) by appropriate "correction factors" $C(\ell_2-\ell_1,p_1,p_2,g)$ with the property that $C = 1+\mathcal{O}(g^6)$ for $\ell_2 > \ell_1 + 3$:

$$\begin{aligned}
\Psi_{\mathcal{U}\mathcal{X}}(\ell_1,\ell_2) = {}& A_{\mathcal{U}\mathcal{X}}\, C_{\mathcal{U}\mathcal{X}}(\ell_2-\ell_1,p_1,p_2,g)\, e^{ip_1\ell_1+ip_2\ell_2} +\\
&+ A'_{\mathcal{U}\mathcal{X}}\, C'_{\mathcal{U}\mathcal{X}}(\ell_2-\ell_1,p_1,p_2,g)\, e^{ip_2\ell_1+ip_1\ell_2}\,,\\
\Psi_{\mathcal{X}\mathcal{U}}(\ell_1,\ell_2) = {}& A_{\mathcal{X}\mathcal{U}}\, C_{\mathcal{X}\mathcal{U}}(\ell_2-\ell_1,p_1,p_2,g)\, e^{ip_1\ell_1+ip_2\ell_2} +\\
&+ A'_{\mathcal{X}\mathcal{U}}\, C'_{\mathcal{X}\mathcal{U}}(\ell_2-\ell_1,p_1,p_2,g)\, e^{ip_2\ell_1+ip_1\ell_2}\,.
\end{aligned} \tag{3.18}$$

This is the perturbative asymptotic Bethe ansatz (PABA) proposed in [18]. A choice[15] which allows to satisfy the necessary consistency conditions in the interaction region $\ell_1 + 4 > \ell_2 > \ell_1$ is

$$\begin{aligned}
C_{\mathcal{U}\mathcal{X}}(\ell_2-\ell_1,p_1,p_2,g) = 1 &+ C^{(2)}_{\mathcal{U}\mathcal{X}}(\ell_2-\ell_1,p_1,p_2)\, g^{2(\ell_2-\ell_1)} +\\
&+ C^{(4)}_{\mathcal{U}\mathcal{X}}(\ell_2-\ell_1,p_1,p_2)\, g^{2+2(\ell_2-\ell_1)} + \mathcal{O}(g^6)\,.
\end{aligned} \tag{3.19}$$

with analogous expressions for the remaining three factors (with either $\mathcal{U} \leftrightarrow \mathcal{X}$, or $C \leftrightarrow C'$, or both). The only role of the correction factors is to correctly disentangle short-range effects from the actual long-range phase shifts determining (3.17).

[14] More generally the asymptotic region is $\ell_2 > \ell_1 + \ell$, where ℓ is the order of perturbation theory.

[15] As in the case of the $\mathfrak{su}(2)$ sector [42] we need a slightly more general three-loop ansatz compared to the $\mathfrak{su}(1|1)$ sector initially treated in [18], where it turned out that $C^{(2)}(1,p1,p2) = C^{(2)}(2,p1,p2)$.

After a significant amount of algebra one finds from [27] for the two-loop correction factors

$$\begin{aligned}
C^{(2)}_{\mathcal{U}\mathcal{X}}(1,p_1,p_2) &= \tfrac{1}{4}\,(e^{-ip_2}-1)(1+4\,i\,\gamma_1-2\,e^{ip_1})\,,\\
C'^{(2)}_{\mathcal{U}\mathcal{X}}(1,p_1,p_2) &= C^{(2)}_{\mathcal{U}\mathcal{X}}(1,p_2,p_1)\,,\\
C^{(2)}_{\mathcal{X}\mathcal{U}}(1,p_1,p_2) &= C^{(2)}_{\mathcal{U}\mathcal{X}}(1,-p_2,-p_1)\,,\\
C'^{(2)}_{\mathcal{X}\mathcal{U}}(1,p_1,p_2) &= C^{(2)}_{\mathcal{U}\mathcal{X}}(1,-p_1,-p_2)\,.
\end{aligned} \tag{3.20}$$

Here γ_1 is one of the three gauge parameters appearing in the two-loop vertex. The two-loop transmission amplitudes are found to be

$$\begin{aligned}
T^{\mathcal{U}\mathcal{X}}_{\mathcal{U}\mathcal{X}}(p_1,p_2) &= \frac{e^{ip_1}-e^{ip_2}}{e^{ip_1+ip_2}-2e^{ip_2}+1}\\
&\quad+\frac{(1-e^{ip_1})(1-e^{ip_2})^2(3-e^{2ip_1}-2e^{-ip_1+ip_2}-e^{ip_1-ip_2}+e^{ip_1+ip_2})}{2(e^{ip_1+ip_2}-2e^{ip_2}+1)^2}\,g^2\\
&\quad+\mathcal{O}(g^4)\,,\\
T^{\mathcal{X}\mathcal{U}}_{\mathcal{X}\mathcal{U}}(p_1,p_2) &= \frac{e^{ip_1}-e^{ip_2}}{e^{ip_1+ip_2}-2e^{ip_2}+1}\\
&\quad+\frac{(1-e^{ip_1})^2(1-e^{ip_2})(3e^{ip_2}-e^{ip_1}+e^{-ip_1}-e^{-ip_2}-2e^{-ip_1+2ip_2})}{2(e^{ip_1+ip_2}-2e^{ip_2}+1)^2}\,g^2\\
&\quad+\mathcal{O}(g^4)\,,
\end{aligned} \tag{3.21}$$

while the two-loop reflection amplitudes are

$$\begin{aligned}
R^{\mathcal{U}\mathcal{X}}_{\mathcal{X}\mathcal{U}}(p_1,p_2) &= -\frac{(1-e^{ip_1})(1-e^{ip_2})}{e^{ip_1+ip_2}-2e^{ip_2}+1}\,\Big(1-(i\gamma_1+\tfrac{1}{4})(e^{ip_1}+e^{-ip_1}-e^{ip_2}-e^{-ip_2})\,g^2\Big)\\
&\quad-\frac{(1-e^{ip_1})(1-e^{ip_2})^2}{2(e^{ip_1+ip_2}-2e^{ip_2}+1)^2}(1+2e^{-ip_1}-2e^{ip_1}+e^{2ip_1}-2e^{-ip_1+ip_2}+\\
&\qquad\qquad+e^{ip_1-ip_2}-2e^{-ip_2}+2e^{ip_2}-e^{ip_1+ip_2})\,g^2\\
&\quad+\mathcal{O}(g^4)\,,\\
R^{\mathcal{X}\mathcal{U}}_{\mathcal{U}\mathcal{X}}(p_1,p_2) &= -\frac{(1-e^{ip_1})(1-e^{ip_2})}{e^{ip_1+ip_2}-2e^{ip_2}+1}\,\Big(1+(i\gamma_1+\tfrac{1}{4})(e^{ip_1}+e^{-ip_1}-e^{ip_2}-e^{-ip_2})\,g^2\Big)\\
&\quad+\frac{(1-e^{ip_1})^2(1-e^{ip_2})}{2(e^{ip_1+ip_2}-2e^{ip_2}+1)^2}(2+e^{-ip_1}-e^{ip_1}-2e^{-ip_1+ip_2}+2e^{ip_1+ip_2}+\\
&\qquad\qquad+2e^{-ip_1+2ip_2}-e^{-ip_2}-e^{ip_2}-2e^{2ip_2})\,g^2\\
&\quad+\mathcal{O}(g^4)\,.
\end{aligned} \tag{3.22}$$

We have also explicitly obtained, using computer algebra, the three-loop correction factors $C^{(2)}(2,p_1,p_2)$, $C^{(4)}(1,p_1,p_2)$ as well as the three-loop modifications of the S-matrix elements, i.e. the $\mathcal{O}(g^4)$ corrections to $T^{\mathcal{U}\mathcal{X}}_{\mathcal{U}\mathcal{X}}$,$T^{\mathcal{X}\mathcal{U}}_{\mathcal{X}\mathcal{U}}$ in (3.21) and $R^{\mathcal{U}\mathcal{X}}_{\mathcal{X}\mathcal{U}}$,$R^{\mathcal{X}\mathcal{U}}_{\mathcal{U}\mathcal{X}}$ in (3.22). The results are too long to display here. They are however crucial for checking our conjectures for the asymptotic all-loop S-matrices in the $\mathfrak{su}(1|2)$ sectors of gauge and string theory in the following Section 3.3.

3.3 All-Loop Factorized S-Matrix for $\mathfrak{su}(1|2)$: A Conjecture

Armed with the perturbative three-loop results for the S-matrix we may now extend the all-loop conjectures of Sec. 2 in an informed fashion to $\mathfrak{su}(1|2)$. The first thing to notice about the amplitudes we found is that neither the two transmission amplitudes (3.21) nor the two reflection amplitudes (3.22) are, respectively, identical, in contradistinction to the one-loop case (3.16). Let us first focus on transmission. As already mentioned at the end of Sec. 3.1, in x-space there are three natural combinations which reduce to the numerator $u_k - u_j$ of the one-loop transmission amplitude. Expanding these to three-loop order and changing variables to momenta p_k, p_j one finds, excitingly, that choosing for the numerators of the amplitudes the two combinations $x_k^+ - x_j^+$ and $x_k^- - x_j^-$ exactly reproduces, respectively, the highly involved three-loop expressions for $T_{\mathcal{UX}}^{\mathcal{UX}}$ and $T_{\mathcal{XU}}^{\mathcal{XU}}$, as displayed (to two loops) in (3.21).

Turning our attention to reflection, we see that the amplitudes are not only asymmetric, but also depend on the gauge parameters. They comprise some of the undetermined coefficients of the Hamiltonian which correspond to similarity transformations. These ambiguities are inevitable in any renormalization scheme. To proceed, observe that at one loop the eigenvalues of the fermion-boson block

$$\begin{pmatrix} T_{\mathcal{UX}}^{\mathcal{UX}} & R_{\mathcal{XU}}^{\mathcal{UX}} \\ \\ R_{\mathcal{UX}}^{\mathcal{XU}} & T_{\mathcal{XU}}^{\mathcal{XU}} \end{pmatrix} \tag{3.23}$$

are precisely the S-matrix elements $S_{\mathcal{XX}}^{\mathcal{XX}}$ and $S_{\mathcal{UU}}^{\mathcal{UU}}$. One may check that this remains true at two and three loops. Assuming this to be true at any order, and using our just presented conjecture for transmission, one finds that

$$(R_{\mathcal{UX}}^{\mathcal{XU}})_{k,j}\,(R_{\mathcal{XU}}^{\mathcal{UX}})_{k,j} = \frac{S_0^2(x_k, x_j)}{(x_k^- - x_j^+)^2}\,(x_k^+ - x_k^-)(x_j^+ - x_j^-)\,. \tag{3.24}$$

We now observe that the combinations $x_k^+ - x_k^-$ and $x_j^+ - x_j^-$ reduce at one loop precisely to the numerators i of the reflection amplitudes, cf. (3.16). After some experimenting, and taking into account (3.24) one finds that the combinations $(x_j^+ - x_j^-)\,\omega_k/\omega_j$ and $(x_k^+ - x_k^-)\,\omega_j/\omega_k$ reproduce, respectively, the two- and three-loop numerators of the reflection amplitudes $R_{\mathcal{XU}}^{\mathcal{UX}}$ and $R_{\mathcal{UX}}^{\mathcal{XU}}$. The gauge parameters appearing in the reflection amplitudes may be absorbed in the functions ω_k. To two loops, cf. (3.22), one has

$$\omega_k = (x_k^+ - x_k^-)^{1/2+2i\gamma_1} + \mathcal{O}(g^4). \tag{3.25}$$

At three loops, the functional form of the function ω_k becomes more involved if one keeps all gauge parameters.

We are then led to the following form for the all-loop asymptotic S-matrix:

$$S_{k,j}(x_k, x_j) = \frac{S_0(x_k, x_j)}{x_k^- - x_j^+} \begin{pmatrix} x_k^+ - x_j^- & & & \\ & x_k^+ - x_j^+ & (x_j^+ - x_j^-)\,\omega_k/\omega_j & \\ & (x_k^+ - x_k^-)\,\omega_j/\omega_k & x_k^- - x_j^- & \\ & & & x_k^- - x_j^+ \end{pmatrix}. \tag{3.26}$$

The global dressing factor $S_0(x_{2,k}, x_{2,j})$, differing slightly between gauge and string theory, is the same as found in Sec. 2:

$$S_0(x_k, x_j) = \frac{1 - g^2/2x_k^+ x_j^-}{1 - g^2/2x_k^- x_j^+} \, \sigma^2(x_k, x_j) \, . \tag{3.27}$$

Excitingly, this *all-loop* S-matrix still satisfies the Yang-Baxter equation (YBE) (3.13). Note that, unlike in most known solutions of the YBE, our S-matrix (3.26) appears (as far as we can see) to not be expressible in terms of the difference of some suitable spectral parameter.[16]

We expect the spectrum to be independent of the function ω_k, as it merely represents a renormalization of the basis of states of the form

$$|\mathcal{X}\ldots \overset{\overset{k_1}{\downarrow}}{\mathcal{U}} \ldots \overset{\overset{\ldots}{\downarrow}}{\mathcal{U}} \ldots \overset{\overset{k_{K_1}}{\downarrow}}{\mathcal{U}} \ldots \mathcal{X}\rangle \longrightarrow \omega_{k_1} \cdots \omega_{k_{K_1}} |\mathcal{X}\ldots \overset{\overset{k_1}{\downarrow}}{\mathcal{U}} \ldots \overset{\overset{\ldots}{\downarrow}}{\mathcal{U}} \ldots \overset{\overset{k_{K_1}}{\downarrow}}{\mathcal{U}} \ldots \mathcal{X}\rangle. \tag{3.28}$$

Applying a reality condition on the Hamiltonian one can actually eliminate such spurious degrees of freedom from the S-matrix (at least up to three loops). This should then lead to a symmetric S-matrix. From (3.24,3.26) we find the unique result to be

$$\omega_k = \sqrt{x_k^+ - x_k^-} \, . \tag{3.29}$$

To second loop order this corresponds to setting the gauge parameter $\gamma_1 = 0$, as seen from (3.25). Actual computations, however, are simplified by the choice $\omega_k = 1$ which we adopt in the following. We leave it as an exercise for the reader to confirm the independence of all observables on ω_k.

3.4 Nested All-Loop Asymptotic Bethe Ansatz for $\mathfrak{su}(1|2)$

The nested Bethe ansatz was discovered, along with the Yang-Baxter equation (3.13), in a seminal paper by C.-N. Yang [52]. This article is very concisely written and we found it useful to consult with the more detailed accounts [53], and in particular [54].

Bethe equations for the long-range chain are derived as in the one-loop case by imposing periodic boundary conditions on the wave function. Ideally this should be done for the exact, all-loop wave functions, but these are currently not known. What we can do is impose periodicity on the asymptotic wave functions such as (3.17). Correspondingly we may only hope to find *asymptotic* Bethe equations. They are expected to break down when the region of interaction reaches the size of the system and the "asymptotic region" effectively shrinks to zero. In gauge theory this is expected to happen around $\mathcal{O}(g^{2L})$ of perturbation theory.

The all-loop asymptotic Bethe equations are identical in form to the one-loop equations (3.15) and we merely use the all-loop S-matrix (3.26) in place of (3.16). While the

[16] Note that (3.26) actually satisfies the YBE without any assumption on the relation between x^+ and x^-. It would be interesting to know whether (3.26) may be transformed to a known solution of the YBE, or whether it constitutes a hitherto unknown, novel solution.

latter is conjectural starting from four loops, it contains the three-loop gauge theory S-matrix calculated in Sec. 3.3. Our Bethe ansatz is thus expected to properly diagonalize arbitrary $\mathfrak{su}(1|2)$ states of $\mathcal{N}=4$ gauge theory to at least third loop order.

Let us then focus on the matrix Bethe equations

$$\left(\frac{x^+_{2,k}}{x^-_{2,k}}\right)^L |\Psi\rangle = S_{k,k+1}\dots S_{k,K_2}S_{k,1}\dots S_{k,k-1}\cdot|\Psi\rangle\,, \tag{3.30}$$

where we have expressed, as in Sec. 2 momenta p_k via (2.29,2.30) by rapidities $x^\pm_{2,k} := x^\pm_k$. For reasons that will become clear shortly we have added a further index 2 to all rapidities. The two-body S-matrix $S_{k,j} = S_{k,j}(x_{2,k}, x_{2,j})$ is given in (3.26). This matrix eigenvalue equation is to be satisfied simultaneously for *all* $k = 1,\dots,K_2$, i.e. the eigenvector $|\Psi\rangle$ is not allowed to depend on k.

We mentioned in Sec. 3.1 that we should think of $|\Psi\rangle$ as a state in a short spin chain of length K_2. Let us start by picking a vacuum (=reference state) on this chain, say the bosonic fields $\mathcal{X}$. We may then say that, cf. (3.1), that we have a length-K_2 spin chain doped with K_1 "magnons" $\mathcal{U}$. Let us define the reduced two-body scattering operator

$$s_{k,j} = \begin{pmatrix} 1 & & & \\ & (t^{\mathcal{UX}}_{\mathcal{UX}})_{k,j} & (r^{\mathcal{UX}}_{\mathcal{XU}})_{k,j} & \\ & (r^{\mathcal{XU}}_{\mathcal{UX}})_{k,j} & (t^{\mathcal{XU}}_{\mathcal{XU}})_{k,j} & \\ & & & (s^{\mathcal{UU}}_{\mathcal{UU}})_{k,j} \end{pmatrix}, \tag{3.31}$$

with

$$S_{k,j}(x_{2,k}, x_{2,j}) = S_0(x_{2,k}, x_{2,j})\,\frac{x^+_{2,k}-x^-_{2,j}}{x^-_{2,k}-x^+_{2,j}}\,s_{k,j}(x_{2,k}, x_{2,j})\,. \tag{3.32}$$

Defining the common eigenvalue of the reduced many-body scattering operator by

$$\lambda_k = \left(\frac{x^+_{2,k}}{x^-_{2,k}}\right)^L \prod_{\substack{j=1\\ j\neq k}}^{K_2} S_0^{-1}(x_{2,k}, x_{2,j})\,\frac{x^-_{2,k}-x^+_{2,j}}{x^+_{2,k}-x^-_{2,j}}\,, \tag{3.33}$$

we may rewrite (3.30) as

$$\lambda_k|\Psi\rangle = s_{k,k+1}\dots s_{k,K_2}s_{k,1}\dots s_{k,k-1}\cdot|\Psi\rangle\,. \tag{3.34}$$

On the vacuum of the short spin chain the reduced operators $s_{k,j}$ act trivially and we find immediately for all values of k

$$\lambda_k|\mathcal{XX}\dots\mathcal{XX}\rangle = |\mathcal{XX}\dots\mathcal{XX}\rangle\,, \tag{3.35}$$

i.e. that $\lambda_k = 1$, which is nothing but our $\mathfrak{su}(2)$ Bethe equation (2.35) or (2.42).

Let us next solve the one-magnon problem of the short chain. Unfortunately we cannot solve the problem by a common Fourier transform, because our short chain is inhomogeneous and thus not translationally invariant. However, introducing a coordinate-space

wave function ψ_k through

$$|\Psi\rangle = \sum_{1\le k\le K_2} \psi_k \, |\mathcal{X}...\mathcal{X}\overset{\overset{k}{\downarrow}}{\mathcal{U}}\mathcal{X}...\mathcal{X}\rangle \, , \tag{3.36}$$

one may write down and solve recursion relations for the amplitudes ψ_k. Interestingly the necessary calculations and considerations, displayed in great detail in [54], are essentially identical to the one-loop case. The idea is to consider (3.34) and to recursively apply the string of reduced two-body operators $s_{k,k-j}$ from "right to left" to the state $|\Psi\rangle$ and investigate how the latter changes. One then derives the following recursion relation on the amplitudes ψ_k:

$$\frac{\psi_{k-j-1}}{\psi_{k-j}} = \frac{(t^{\mathcal{U}\mathcal{X}}_{\mathcal{U}\mathcal{X}})_{k,k-j}\lambda_k - \Delta_{k,k-j}}{\lambda_k - (t^{\mathcal{X}\mathcal{U}}_{\mathcal{X}\mathcal{U}})_{k,k-j-1}} \, \frac{(r^{\mathcal{X}\mathcal{U}}_{\mathcal{U}\mathcal{X}})_{k,k-j-1}}{(r^{\mathcal{X}\mathcal{U}}_{\mathcal{U}\mathcal{X}})_{k,k-j}} \, , \tag{3.37}$$

where

$$\Delta_{k,k-j} = (t^{\mathcal{U}\mathcal{X}}_{\mathcal{U}\mathcal{X}})_{k,k-j}(t^{\mathcal{X}\mathcal{U}}_{\mathcal{X}\mathcal{U}})_{k,k-j} - (r^{\mathcal{U}\mathcal{X}}_{\mathcal{X}\mathcal{U}})_{k,k-j}(r^{\mathcal{X}\mathcal{U}}_{\mathcal{U}\mathcal{X}})_{k,k-j} \, . \tag{3.38}$$

Now, using (3.26,3.31) for our transmission amplitudes

$$(t^{\mathcal{U}\mathcal{X}}_{\mathcal{U}\mathcal{X}})_{k,k-j} = \frac{x^+_{2,k} - x^+_{2,k-j}}{x^+_{2,k} - x^-_{2,k-j}} \, , \qquad (t^{\mathcal{X}\mathcal{U}}_{\mathcal{X}\mathcal{U}})_{k,k-j} = \frac{x^-_{2,k} - x^-_{2,k-j}}{x^+_{2,k} - x^-_{2,k-j}} \, , \tag{3.39}$$

and reflection amplitudes

$$(r^{\mathcal{X}\mathcal{U}}_{\mathcal{U}\mathcal{X}})_{k,k-j} = \frac{x^+_{2,k} - x^-_{2,k}}{x^+_{2,k} - x^-_{2,k-j}} \, , \qquad (r^{\mathcal{U}\mathcal{X}}_{\mathcal{X}\mathcal{U}})_{k,k-j} = \frac{x^+_{2,k-j} - x^-_{2,k-j}}{x^+_{2,k} - x^-_{2,k-j}} \, , \tag{3.40}$$

we may calculate the determinant $\Delta_{k,k-j}$ to be

$$\Delta_{k,k-j} = \frac{x^-_{2,k} - x^+_{2,k-j}}{x^+_{2,k} - x^-_{2,k-j}} = (s^{\mathcal{U}\mathcal{U}}_{\mathcal{U}\mathcal{U}})_{k,k-j} \, , \tag{3.41}$$

i.e. it happens to coincide with the reduced fermion-fermion scattering amplitude. We may then rewrite (3.37) as

$$\frac{\psi_{k-j-1}}{\psi_{k-j}} = \frac{x^+_{2,k-j} - \frac{x^+_{2,k}\lambda_k - x^-_{2,k}}{\lambda_k - 1}}{x^-_{2,k-j-1} - \frac{x^+_{2,k}\lambda_k - x^-_{2,k}}{\lambda_k - 1}} \, . \tag{3.42}$$

The left hand side of this equation only depends on the difference $k-j$. Therefore, the equation can only be consistent if the right hand side does *not* depend on the index k. We then conclude that

$$x_1 := \frac{x^+_{2,k}\lambda_k - x^-_{2,k}}{\lambda_k - 1} \tag{3.43}$$

must be a constant. Iterating (3.42) one finds the following elegant result for the one-magnon wave functions

$$\psi_k(x_1) = \prod_{j=1}^{k-1} \frac{x_1 - x_{2,j}^-}{x_1 - x_{2,j+1}^+} \,. \tag{3.44}$$

We may interpret x_1 as a new rapidity variable, parametrizing the "momentum" of our magnon $\mathcal{U}$. One may also check explicitly, with some work [53], that the wave function (3.44) we just derived indeed satisfies the matrix eigenvalue equation (3.34). The eigenvalue is found from inverting (3.43)

$$\lambda_k = \lambda_k(x_1) := \frac{x_{2,k}^- - x_1}{x_{2,k}^+ - x_1} \,. \tag{3.45}$$

Is the value of the magnon rapidity x_1 arbitrary? It turns out that it is not: Demanding periodic boundary conditions for our short chain leads to the quantization condition

$$\prod_{k=1}^{K_2} \lambda_k = 1 \,. \tag{3.46}$$

This completely solves the one-magnon problem $K_1 = 1$.

Next we will study the matrix eigenvalue equation (3.34) for the two-magnon problem $K_1 = 2$. The states are now described by coordinate-space wave functions ψ_{k_1,k_2} through

$$|\Psi\rangle = \sum_{1 \le k_1 < k_2 \le K_2} \psi_{k_1,k_2} \, |\mathcal{X} \ldots \mathcal{X} \overset{\overset{k_1}{\downarrow}}{\mathcal{U}} \mathcal{X} \ldots \mathcal{X} \overset{\overset{k_2}{\downarrow}}{\mathcal{U}} \mathcal{X} \ldots \mathcal{X}\rangle \,, \tag{3.47}$$

Denoting the level-two rapidities of the two magnons by $x_{1,1}$ and $x_{1,2}$, a first guess for the solution would be that the one-magnon eigenvalue of (3.45) is now replaced by the product

$$\lambda_k = \lambda_k(x_{1,1})\, \lambda_k(x_{1,2}) = \frac{x_{2,k}^- - x_{1,1}}{x_{2,k}^+ - x_{1,1}} \, \frac{x_{2,k}^- - x_{1,2}}{x_{2,k}^+ - x_{1,2}} \,, \tag{3.48}$$

and that the two-body wave function is just the product $\psi_{k_1}(x_{1,1})\, \psi_{k_2}(x_{1,2})$ of one-body amplitudes (3.44). The first part of this guess, (3.48), is indeed correct. However, in order to find the correct wave function we need to take into account the scattering of the particles. If the scattering is non-diffractive, the particles may only exchange their rapidities. It is then reasonable to attempt a (secondary) Bethe Ansatz:[17]

$$\psi_{k_1,k_2}(x_{1,1}, x_{1,2}) = B\, \psi_{k_1}(x_{1,1})\, \psi_{k_2}(x_{1,2}) - B'\, \psi_{k_1}(x_{1,2})\, \psi_{k_2}(x_{1,1}) \,. \tag{3.49}$$

In similarity to the treatment for the original, long spin chain we have to check explicitly that this ansatz indeed satisfies (3.34) with (3.48). Abbreviating, respectively, $\psi_k(x_{1,1})$,$\psi_k(x_{1,2})$ by ψ_k,ψ'_k and $\lambda_k(x_{1,1})$,$\lambda_k(x_{1,2})$ by λ_k,λ'_k we find the consistency condition for factorized scattering to be

$$\frac{B'}{B} = \frac{\Delta_{k,k-j}\, \lambda'_{k-j}\, \psi'_k\, \psi_{k-j}\, \lambda_{k-1} \ldots \lambda_{k-j} + \psi_k\, \psi'_{k-j}\, \lambda'_k \ldots \lambda'_{k-j}}{\Delta_{k,k-j}\, \lambda_{k-j}\, \psi_k\, \psi'_{k-j}\, \lambda'_{k-1} \ldots \lambda'_{k-j} + \psi'_k\, \psi_{k-j}\, \lambda_k \ldots \lambda_{k-j}} \,, \tag{3.50}$$

[17] The sign in front of B' is due to the exchange of two fermions.

where obviously the amplitude Δ for the $\mathcal{U}$-$\mathcal{U}$ exchange enters, see (3.41). Our ansatz requires that the left hand side of (3.50) should neither depend on k nor on j. Now, using (3.41,3.44,3.45) we find

$$\frac{B'}{B} = \frac{(x^+_{k-j} - x^-_k)(x^-_{k-j} - x_{1,2})(x^+_k - x_{1,1}) + (x^-_{k-j} - x^+_k)(x^-_k - x_{1,2})(x^+_{k-j} - x_{1,1})}{(x^+_{k-j} - x^-_k)(x^-_{k-j} - x_{1,1})(x^+_k - x_{1,2}) + (x^-_{k-j} - x^+_k)(x^-_k - x_{1,1})(x^+_{k-j} - x_{1,2})}\,. \tag{3.51}$$

After a short computation one ends up with the remarkably simple result

$$\frac{B'}{B} = 1\,. \tag{3.52}$$

We see that the unwanted dependence on the indices k and j has disappeared and that the secondary Bethe ansatz (3.49) yields indeed the correct two-magnon wave function with eigenvalue (3.48).

In general we do not have to find the wave function of the short chain explicitly, but merely impose periodic boundary conditions. Due to the simplicity of (3.52), this is easily done here: Note that the two-magnon wave function is of the form of a Slater determinant:

$$\psi_{k_1,k_2}(x_{1,1}, x_{1,2}) = \psi_{k_1}(x_{1,1})\,\psi_{k_2}(x_{1,2}) - \psi_{k_1}(x_{1,2})\,\psi_{k_2}(x_{1,1})\,, \tag{3.53}$$

indicating that the fermions $\mathcal{U}$ in the short, inhomogeneous chain are "free" to all loop orders. And indeed, since scattering is factorized in the short chain, we may immediately write down the K_1-magnon wave functions as a $K_1 \times K_1$ Slater determinant:

$$\psi_{k_1,\ldots,k_{K_1}}(x_{1,1}, \ldots, x_{1,K_1}) = \det_{\mu,\nu} \psi_{k_\mu}(x_{1,\nu})\,, \tag{3.54}$$

To prevent confusion let us reiterate that the indices μ, ν are labels for the K_1 magnons in the auxiliary, short spin chain of length K_2. The indices k_μ indicate the position of these magnons in the auxiliary chain and therefore take values in the set $\{1, \ldots, K_2\}$. Likewise, the $x_{1,\mu}$ are the rapidities of the K_1 magnons describing their motion in the short chain of length K_2. They are not to be confused with the rapidities $x_{2,k}$ describing the motion of the original K_2 magnons in the long chain of length L. Note that the one-body wave functions $\psi_{k_\mu}(x_{1,\nu})$ in (3.54,3.53) depend implicitly also on the magnon rapidities $x_{2,k}$ of the original chain, as seen from the result (3.44).

The eigenvalue associated to the wave function (3.54) is clearly given by the product $\lambda_k = \lambda_k(x_{1,1}) \ldots \lambda_k(x_{1,K_1})$, in generalization of the one and two-magnon expressions (3.45,3.48). Using (3.33), we thus derived the first set of asymptotic Bethe equations for the $\mathfrak{su}(1|2)$ sector. They read, for $k = 1, \ldots, K_2$

$$\left(\frac{x^+_{2,k}}{x^-_{2,k}}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{K_2} S_0(x_{2,k}, x_{2,j}) \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{x^+_{2,k} - x^-_{2,j}}{x^-_{2,k} - x^+_{2,j}} \prod_{j=1}^{K_1} \frac{x^-_{2,k} - x_{1,j}}{x^+_{2,k} - x_{1,j}}\,. \tag{3.55}$$

Finally we need to impose periodic boundary conditions on the small chain. Generalizing the one-magnon case (3.46), this yields a second set of Bethe equations. Taking each of

the $k = 1, \ldots, K_1$ particles once around the chain of length K_2, we then find (recall that the fermions are free)

$$1 = \prod_{j=1}^{K_2} \frac{x_{1,k} - x_{2,j}^+}{x_{1,k} - x_{2,j}^-}\,, \tag{3.56}$$

which is nothing but an inhomogenous version of the free particle quantization law.

We could alternatively have picked the fermions $\mathcal{U}$ as a reference state in the short spin chain of length K_2. We should then consider the $\tilde{K}_1$ scalars $\mathcal{X}$ as excitations on this vacuum state:

$$\mathrm{Tr}\ \mathcal{X}^{\tilde{K}_1} \mathcal{U}^{K_2 - \tilde{K}_1} \mathcal{Z}^{L-K_2} + \ldots\,. \tag{3.57}$$

In this case one should replace the reduced two-body scattering operator (3.31) by

$$\tilde{s}_{k,j} = \begin{pmatrix} (\tilde{s}^{\mathcal{X}\mathcal{X}}_{\mathcal{X}\mathcal{X}})_{k,j} & & & \\ & (\tilde{t}^{\mathcal{U}\mathcal{X}}_{\mathcal{U}\mathcal{X}})_{k,j} & (\tilde{r}^{\mathcal{U}\mathcal{X}}_{\mathcal{X}\mathcal{U}})_{k,j} & \\ & & & \\ & (\tilde{r}^{\mathcal{X}\mathcal{U}}_{\mathcal{U}\mathcal{X}})_{k,j} & (\tilde{t}^{\mathcal{X}\mathcal{U}}_{\mathcal{X}\mathcal{U}})_{k,j} & \\ & & & 1 \end{pmatrix}, \tag{3.58}$$

and express the S-matrix entering the matrix Bethe equation (3.30) as

$$S_{k,j}(x_{2,k}, x_{2,j}) = S_0(x_{2,k}, x_{2,j})\, \tilde{s}_{k,j}(x_{2,k}, x_{2,j})\,. \tag{3.59}$$

The further analysis proceeds in exactly the way we just presented for the bosonic vacuum and it is straightforward to obtain the expressions for the eigenvalues and wave functions. In particular the magnons $\mathcal{X}$ again behave like free particles in the reduced chain. We will just state the corresponding Bethe equations which read, for $k = 1, \ldots, K_2$

$$\left(\frac{x_{2,k}^+}{x_{2,k}^-}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{K_2} S_0(x_{2,k}, x_{2,j}) \prod_{j=1}^{\tilde{K}_1} \frac{x_{2,k}^+ - \tilde{x}_{1,j}}{x_{2,k}^- - \tilde{x}_{1,j}}\,, \tag{3.60}$$

where $S_0(x_{2,k}, x_{2,j})$ is the same long-range scattering matrix (3.27) as in the first form (3.55) of the Bethe ansatz. The second set of equations is, with $k = 1, \ldots, \tilde{K}_1$

$$1 = \prod_{j=1}^{K_2} \frac{\tilde{x}_{1,k} - x_{2,j}^-}{\tilde{x}_{1,k} - x_{2,j}^+}\,. \tag{3.61}$$

Later in Section 4.7 we will see, in much greater generality, that this second form (3.60,3.61) of the Bethe equations may also be obtained directly from the first form (3.55,3.56) by a duality transformation [49, 55, 14].

3.5 Spectrum

We have computed the spectrum of all states of the $\mathfrak{su}(1|2)$ spin chain with $L \leq 8$ using the three-loop Hamiltonian found in [27]. The results are presented in Tab. 4 excluding

L	K_2	K_1	$(E_0, E_2, E_{4\mathrm{g}} \vert E_{4\mathrm{s}})^P$
7	4	2	$(10, -\frac{75}{4}, \frac{4315}{64} \vert \frac{3963}{64})^\pm$ $\surd$
			$(6, -\frac{33}{4}, \frac{1557}{64} \vert \frac{1461}{64})^\pm$ $\surd$
	5	3	$(12, -\frac{45}{2}, \frac{1281}{16} \vert \frac{1137}{16})^+$
			$(8, -\frac{25}{2}, \frac{687}{16} \vert \frac{655}{16})^+$
8	4	2	$\big(31E^3-350E^2+1704E-3016,\ -50E^3+1111E^2-7971E+18452,$ $\frac{337}{2}E^3-\frac{18363}{4}E^2+38740E-102390 \,\big\vert\, \frac{621}{4}E^3-\frac{17213}{4}E^2+36730E-97840\big)^\pm$
	5	2	$(8, -13, \frac{343}{8} \vert \frac{311}{8})^\pm$
			$(15E-48, -23E+135, \frac{595}{8}E-\frac{4023}{8} \vert \frac{541}{8}E-\frac{3735}{8})^\pm$
	5	3	$(7, -\frac{21}{2}, \frac{3241}{94} \vert \frac{6083}{188})^\pm$
			$\big(33E^2-358E+1279,\ -\frac{119}{2}E^2+1283E-\frac{13675}{2},$ $\frac{19601}{94}E^2-\frac{1059061}{188}E+\frac{1693257}{47} \,\big\vert\, \frac{8784}{47}E^2-\frac{972991}{188}E+\frac{6323573}{188}\big)^\pm$
	6	4	$(12, -22, 78 \vert 72)^\pm$

Table 4: Spectrum of lowest-lying states genuinely in the $\mathfrak{su}(1|2)$ sector.

those which have been given before in Tab. 1,2,3. The notation is explained in Sec. 2.2. The states marked as "$\surd$" have been computed using the Bethe equations (3.55,3.56). Note that the state with one-loop energy $E_0 = 6$ is singular, i.e. it has a pair of roots of type 2 at $u_2 = \pm\frac{i}{2} + \delta u$, where $\delta u = \mathcal{O}(g^2)$ is the same for both roots. We kindly invite the reader to confirm that the energies of the remaining cases also agree with our nested Bethe ansatz.

Note that the excitation numbers K_2, K_1 correspond to the Dynkin labels $[q_1, p, q_2]$ of $\mathfrak{su}(4)$

$$q_1 = K_2 - K_1, \qquad p = L + K_1 - 2K_2, \qquad q_2 = K_2 \tag{3.62}$$

and the labels $[s_1, r, s_2]$ of $\mathfrak{su}(2,2)$

$$s_1 = K_1, \qquad r = -L - K_1 - 2K_2 - \delta D, \qquad s_2 = 0. \tag{3.63}$$

4 The Factorized S-Matrix of the $\mathfrak{su}(1,1|2)$ Sector

In the previous Section 3 we have unified the bosonic $\mathfrak{su}(2)$ and fermionic $\mathfrak{su}(1|1)$ sectors into a supersymmetric, long-range t-J model with $\mathfrak{su}(1|2)$ symmetry. In turn, in [18] it was demonstrated that the integrable structure of the non-compact bosonic $\mathfrak{sl}(2)$ sector is also closely related to the $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$ sectors, confer (1.2) and the discussion in Sec. 2. It is therefore very natural to attempt to find the S-matrix and the associated Bethe ansatz for the smallest closed sector unifying all three two-component sectors. This is the $\mathfrak{su}(1,1|2)$ sector (see [25]). In addition to the scalars $\mathcal{Z}, \mathcal{X}$, the derivative $\mathcal{D}$ and the gaugino $\mathcal{U}$, it requires a second fermion $\dot{\mathcal{U}}$. The possible states at a given lattice site are then

$$\mathcal{D}^k\mathcal{Z}, \quad \mathcal{D}^k\mathcal{X}, \quad \mathcal{D}^k\mathcal{U}, \quad \mathcal{D}^k\dot{\mathcal{U}} \tag{4.1}$$

where k may be any non-negative integer. We consider $\mathcal{Z}$ as the vacuum and its single excitations are

$$\mathcal{Z} \to \mathcal{X}, \quad \mathcal{Z} \to \mathcal{U}, \quad \mathcal{Z} \to \dot{\mathcal{U}}, \quad \mathcal{Z} \to \mathcal{D}\mathcal{Z}. \tag{4.2}$$

Note that the scalar $\mathcal{Z} \to \mathcal{X}$ is a hard-core excitation, there can be only one such excitation per site. Conversely, the derivative $\mathcal{Z} \to \mathcal{DZ}$ is a soft-core excitation, there can be arbitrarily many excitations per site and they also exist on sites which are already occupied by scalars or fermions. In this context it is useful to consider the fermions as excitation with a mixed type of core: A fermion cannot coexist with a scalar or another fermion of the same type. A mixture of the two fermions $\mathcal{Z} \to \mathcal{U}$ and $\mathcal{Z} \to \dot{\mathcal{U}}$ however is possible and should be considered as the double excitation $\mathcal{Z} \to \mathcal{DX}$ in agreement with supersymmetry transformation rules.

This is the largest sector where the spin chain remains "static", i.e. the length does not fluctuate [27]. We are therefore still on firm grounds and should be able to apply the technology established in the previous section to this extended non-compact sector. However, it still exhibits an exciting new feature as compared to the previously discussed cases. The interactions allow for *flavor change*:

$$\mathcal{U}\dot{\mathcal{U}} \leftrightarrow \mathcal{X}\,(\mathcal{DZ})\,. \tag{4.3}$$

This means that a pair of fermions may annihilate and produce a pair of bosons. While particle annihilation and production are often believed to destroy integrability, here the number of excitations is actually preserved by the flavor change: This is related to the above claim that a combination of the two excitations $\mathcal{Z} \to \mathcal{U}$ and $\mathcal{Z} \to \dot{\mathcal{U}}$ is equivalent to the double excitation $\mathcal{Z} \to \mathcal{DX}$. Among the above four single excitations there is one linear dependence (the sector has rank 3) which allows for the flavor change.

As in [18] we are currently unable to derive the higher loop S-matrix by the PABA in this sector since the dilatation operator is not known beyond one loop. However, we may inspect the one-loop S-matrix in this sector and generalize it to all loops according to the principles discovered above. Excitingly, we shall find that the result still satisfies the Yang-Baxter equation! In addition, our conjecture is consistent with multiplet splitting, dualization and the expected thermodynamics.

4.1 One-Loop Scattering in the $\mathfrak{su}(1,1|2)$ Sector

Let us consider the vacuum state

$$|0\rangle = |\mathcal{Z} \ldots \mathcal{Z}\rangle \tag{4.4}$$

which has zero energy. An eigenstate with a single excitation $\mathcal{A} = \mathcal{X}, \mathcal{U}, \dot{\mathcal{U}}, \mathcal{DZ}$ is given by

$$|p_{\mathcal{A}}\rangle = \sum_{\ell} e^{ip\ell}\, |\mathcal{Z} \ldots \mathcal{Z}\overset{\overset{\ell}{\downarrow}}{\mathcal{A}}\mathcal{Z} \ldots \mathcal{Z}\rangle \tag{4.5}$$

Its energy eigenvalue is $e_0(p) = 4\sin^2(p/2)$. We will now represent the states in a slightly different fashion using oscillator excitations

$$|p_{\mathbf{A},\dot{\mathbf{A}}}\rangle = \sum_{\ell} e^{ip\ell}\mathbf{A}^\dagger_\ell \dot{\mathbf{A}}^\dagger_\ell |0\rangle. \tag{4.6}$$

We have replaced the field $\mathcal{A}$ by a pair of harmonic oscillators $\mathbf{A} = \mathbf{a}, \mathbf{c}$ and $\dot{\mathbf{A}} = \dot{\mathbf{a}}, \dot{\mathbf{c}}$ acting on the vacuum at site ℓ. The oscillators $\mathbf{a}, \dot{\mathbf{a}}$ are bosonic while $\mathbf{c}, \dot{\mathbf{c}}$ are fermionic. The four different elementary excitations $\mathcal{A} = \mathcal{X}, \mathcal{U}, \dot{\mathcal{U}}, \mathcal{DZ}$ are represented by a combination of one $\mathbf{A}^\dagger$ and one $\dot{\mathbf{A}}^\dagger$

$$|\mathcal{X}\rangle = \mathbf{c}^\dagger \dot{\mathbf{c}}^\dagger |\mathcal{Z}\rangle, \qquad |\mathcal{U}\rangle = \mathbf{a}^\dagger \dot{\mathbf{c}}^\dagger |\mathcal{Z}\rangle, \qquad |\dot{\mathcal{U}}\rangle = \mathbf{c}^\dagger \dot{\mathbf{a}}^\dagger |\mathcal{Z}\rangle, \qquad |\mathcal{DZ}\rangle = \mathbf{a}^\dagger \dot{\mathbf{a}}^\dagger |\mathcal{Z}\rangle. \tag{4.7}$$

In this notation we will be able to write the scattering eigenstate of two excitations in a very concise form. The eigenstate is defined by the equation

$$\mathcal{H}_0 |p_{\mathbf{A},\dot{\mathbf{A}}}; q_{\mathbf{B},\dot{\mathbf{B}}}\rangle = \big(e_0(p) + e_0(q)\big) |p_{\mathbf{A},\dot{\mathbf{A}}}; q_{\mathbf{B},\dot{\mathbf{B}}}\rangle \tag{4.8}$$

and the boundary condition that the wave function is a product of two instances of (4.6) when the excitations are far apart. We find the following set of independent states specified by the momenta p, q and the oscillators $\mathbf{A}, \dot{\mathbf{A}}, \mathbf{B}, \dot{\mathbf{B}}$

$$\begin{aligned}
|p_{\mathbf{A},\dot{\mathbf{A}}}; q_{\mathbf{B},\dot{\mathbf{B}}}\rangle = &\sum_{\ell_1<\ell_2} e^{ip\ell_1+iq\ell_2} \mathbf{A}^\dagger_{\ell_1} \dot{\mathbf{A}}^\dagger_{\ell_1} \mathbf{B}^\dagger_{\ell_2} \dot{\mathbf{B}}^\dagger_{\ell_2} |0\rangle \\
&+ \sum_{\ell_1=\ell_2} \frac{u-v}{u-v-i}\, e^{ip\ell_1+iq\ell_2} \mathbf{A}^\dagger_{\ell_1} \dot{\mathbf{A}}^\dagger_{\ell_1} \mathbf{B}^\dagger_{\ell_1} \dot{\mathbf{B}}^\dagger_{\ell_1} |0\rangle \\
&+ \sum_{\ell_1>\ell_2} \frac{(u-v)^2}{(u-v-i)(u-v+i)}\, e^{ip\ell_1+iq\ell_2} \mathbf{A}^\dagger_{\ell_1} \dot{\mathbf{A}}^\dagger_{\ell_1} \mathbf{B}^\dagger_{\ell_2} \dot{\mathbf{B}}^\dagger_{\ell_2} |0\rangle \\
&+ \sum_{\ell_1>\ell_2} \frac{i(u-v)}{(u-v-i)(u-v+i)}\, e^{ip\ell_1+iq\ell_2} \mathbf{A}^\dagger_{\ell_2} \dot{\mathbf{A}}^\dagger_{\ell_1} \mathbf{B}^\dagger_{\ell_1} \dot{\mathbf{B}}^\dagger_{\ell_2} |0\rangle \\
&+ \sum_{\ell_1>\ell_2} \frac{i(u-v)}{(u-v-i)(u-v+i)}\, e^{ip\ell_1+iq\ell_2} \mathbf{A}^\dagger_{\ell_1} \dot{\mathbf{A}}^\dagger_{\ell_2} \mathbf{B}^\dagger_{\ell_2} \dot{\mathbf{B}}^\dagger_{\ell_1} |0\rangle \\
&+ \sum_{\ell_1>\ell_2} \frac{i^2}{(u-v-i)(u-v+i)}\, e^{ip\ell_1+iq\ell_2} \mathbf{A}^\dagger_{\ell_2} \dot{\mathbf{A}}^\dagger_{\ell_2} \mathbf{B}^\dagger_{\ell_1} \dot{\mathbf{B}}^\dagger_{\ell_1} |0\rangle.
\end{aligned} \tag{4.9}$$

The rapidities u, v are defined via

$$e^{ip} = \frac{u+\frac{i}{2}}{u-\frac{i}{2}}\,, \qquad e^{iq} = \frac{v+\frac{i}{2}}{v-\frac{i}{2}}\,. \tag{4.10}$$

The first line represents the incoming excitations. The second line represents the wave-function when the two excitations overlap and is only present for particles which experience soft-core scattering. The remaining four lines represent outgoing excitations and they encode the S-matrix.

Let us explain how to obtain the S-matrix from the last four lines in more detail. The simplest case is when all four oscillators $\mathbf{A}, \dot{\mathbf{A}}, \mathbf{B}, \dot{\mathbf{B}}$ are different. In that case flavor changes are possible because the excitations $\mathbf{A}, \dot{\mathbf{A}}$ and $\mathbf{B}, \dot{\mathbf{B}}$ can recombine as $\mathbf{A}, \dot{\mathbf{B}}$ and $\mathbf{B}, \dot{\mathbf{A}}$ (middle two lines). If no flavor change takes place, there can either be transmission (first line) or reflection (last line). Note that despite the recombinations we have not changed the order of oscillators in (4.9) because some of the oscillators can be fermionic.

This would lead to various additional signs which are not necessary in the form (4.9). When two of the oscillators are equal, say $\mathbf{A} = \mathbf{B}$, there can only be transmission and reflection. The contributions to the S-matrix elements now come from adding two lines, the first two (transmission) or the latter two (reflection). Here the statistics of oscillators determines the outcome of the sum: It can either yield a factor of $u - v + i$ or $u - v - i$, both of which will cancel against one of the denominators and yield elements of the sort (3.16). If in addition $\dot{\mathbf{A}} = \dot{\mathbf{B}}$, both excitations are equal and all four lines combine according to statistics.

In fact, the scattering state (4.9) is completely general for any unitary algebra $\mathfrak{sl}(m|n)$ with the spin sites in oscillator representations. In particular it applies to the complete $\mathcal{N} = 4$ one-loop spin chain [33,3]. There the excitations are made out of two sets of four (instead of two) oscillators $\mathbf{A} = \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ and $\dot{\mathbf{A}} = \dot{\mathbf{a}}, \dot{\mathbf{b}}, \dot{\mathbf{c}}, \dot{\mathbf{d}}$. The oscillators $\mathbf{a}, \mathbf{b}, \dot{\mathbf{a}}, \dot{\mathbf{b}}$ are bosonic whereas $\mathbf{c}, \mathbf{d}, \dot{\mathbf{c}}, \dot{\mathbf{d}}$ as fermionic. The excitations are made up from a pair of oscillators $\mathbf{A}, \dot{\mathbf{A}}$ and consequently there are 8 bosonic and 8 fermionic excitations in total, cf. [19,14].

4.2 The Asymptotic S-Matrix for $\mathfrak{su}(1,1|2)$

Based on the above complete one-loop S-matrix we make an educated guess of how to extend the all-loop S-matrix from the $\mathfrak{su}(1|2)$ sector to the $\mathfrak{su}(1,1|2)$ sector which also includes the derivative sector $\mathfrak{su}(1,1) = \mathfrak{sl}(2)$ of Sec. 2. As before, the S-matrix should have an overall prefactor S_0 which contains auxiliary scattering terms and which distinguishes between gauge theory and the string chain. The remaining terms have a common denominator which, in agreement with (4.9), is composed from two factors $(x_k^+ - x_j^-)(x_k^- - x_j^+)$. The individual elements of the S-matrix differ only by the numerator N which is also a product of two terms "$x_{k,j}^\pm - x_{k,j}^\pm$"

$$S_{ab}^{cd}(x_k, x_j) = S_0(x_k, x_j)\, \frac{N_{ab}^{cd}(x_k, x_j)}{(x_k^+ - x_j^-)(x_k^- - x_j^+)}\,. \tag{4.11}$$

From (2.39) we can read off the single-flavor scattering terms of scalars, fermions and derivatives (we shall abbreviate the excitation $\mathcal{DZ}$ by just $\mathcal{D}$)

$$\begin{aligned} N_{\mathcal{XX}}^{\mathcal{XX}} &= (x_k^+ - x_j^-)(x_k^+ - x_j^-), & N_{\dot{\mathcal{U}}\dot{\mathcal{U}}}^{\dot{\mathcal{U}}\dot{\mathcal{U}}} &= (x_k^+ - x_j^-)(x_k^- - x_j^+), \\ N_{\mathcal{UU}}^{\mathcal{UU}} &= (x_k^+ - x_j^-)(x_k^- - x_j^+), & N_{\mathcal{DD}}^{\mathcal{DD}} &= (x_k^- - x_j^+)(x_k^- - x_j^+). \end{aligned} \tag{4.12}$$

The scattering of scalars $\mathcal{X}$ and fermions $\mathcal{U}$ we know from (3.26)

$$\begin{aligned} N_{\mathcal{XU}}^{\mathcal{XU}} &= (x_k^+ - x_j^-)(x_k^- - x_j^-), & N_{\mathcal{UX}}^{\mathcal{XU}} &= (x_k^+ - x_j^-)(x_k^+ - x_k^-), \\ N_{\mathcal{XU}}^{\mathcal{UX}} &= (x_k^+ - x_j^-)(x_j^+ - x_j^-), & N_{\mathcal{UX}}^{\mathcal{UX}} &= (x_k^+ - x_j^-)(x_k^+ - x_j^+). \end{aligned} \tag{4.13}$$

Note that we have omitted the factors ω because we have seen in Sec. 3.3 that they are irrelevant for the spectrum. To obtain a symmetric S-matrix one would have to replace both $x_k^+ - x_k^-$ and $x_j^+ - x_j^-$ by their geometric mean $\sqrt{x_k^+ - x_k^-}\sqrt{x_j^+ - x_j^-}$ at all places. As the fermion $\dot{\mathcal{U}}$ should have the same interactions with the scalars as $\mathcal{U}$ we expect the S-matrix elements of $\mathcal{X}$ with $\dot{\mathcal{U}}$ to be given by the same expressions (4.13).

The scattering between fermions and derivatives most likely takes a very similar form but with some changes in the signs (same for $\mathcal{U}$ instead of $\dot{\mathcal{U}}$)

$$\begin{aligned} N^{\dot{\mathcal{U}}\mathcal{D}}_{\dot{\mathcal{U}}\mathcal{D}} &= (x_k^- - x_j^+)(x_k^- - x_j^-), & N^{\dot{\mathcal{U}}\mathcal{D}}_{\mathcal{D}\dot{\mathcal{U}}} &= (x_k^- - x_j^+)(x_k^- - x_k^+), \\ N^{\mathcal{D}\dot{\mathcal{U}}}_{\dot{\mathcal{U}}\mathcal{D}} &= (x_k^- - x_j^+)(x_j^- - x_j^+), & N^{\mathcal{D}\dot{\mathcal{U}}}_{\mathcal{D}\dot{\mathcal{U}}} &= (x_k^- - x_j^+)(x_k^+ - x_j^+). \end{aligned} \tag{4.14}$$

The largest sector of the S-matrix is the scattering between the scalar $\mathcal{X}$ and the derivative $\mathcal{D}$ which can mix with the scattering of both fermions $\mathcal{U}$ and $\dot{\mathcal{U}}$. Our proposal for the remaining numerators is[18]

$$\begin{aligned} N^{\mathcal{X}\mathcal{D}}_{\mathcal{X}\mathcal{D}} &= (x_k^- - x_j^-)(x_k^- - x_j^-), & N^{\mathcal{X}\mathcal{D}}_{\mathcal{U}\dot{\mathcal{U}}} &= (x_k^- - x_j^-)(x_k^+ - x_k^-), \\ N^{\mathcal{U}\dot{\mathcal{U}}}_{\mathcal{X}\mathcal{D}} &= (x_k^- - x_j^-)(x_j^+ - x_j^-), & N^{\mathcal{U}\dot{\mathcal{U}}}_{\mathcal{U}\dot{\mathcal{U}}} &= (x_k^- - x_j^-)(x_k^+ - x_j^+), \\ N^{\dot{\mathcal{U}}\mathcal{U}}_{\mathcal{X}\mathcal{D}} &= (x_k^- - x_j^-)(x_j^- - x_j^+), & N^{\dot{\mathcal{U}}\mathcal{U}}_{\mathcal{U}\dot{\mathcal{U}}} &= (x_k^- - x_k^+)(x_j^+ - x_j^-), \\ N^{\mathcal{D}\mathcal{X}}_{\mathcal{X}\mathcal{D}} &= (x_j^- - x_j^+)(x_j^- - x_j^+), & N^{\mathcal{D}\mathcal{X}}_{\mathcal{U}\dot{\mathcal{U}}} &= (x_k^+ - x_j^+)(x_j^+ - x_j^-), \\[1ex] N^{\mathcal{X}\mathcal{D}}_{\dot{\mathcal{U}}\mathcal{U}} &= (x_k^- - x_j^-)(x_k^- - x_k^+), & N^{\mathcal{X}\mathcal{D}}_{\mathcal{D}\mathcal{X}} &= (x_k^- - x_k^+)(x_k^- - x_k^+), \\ N^{\mathcal{U}\dot{\mathcal{U}}}_{\dot{\mathcal{U}}\mathcal{U}} &= (x_k^- - x_k^+)(x_j^+ - x_j^-), & N^{\mathcal{U}\dot{\mathcal{U}}}_{\mathcal{D}\mathcal{X}} &= (x_k^+ - x_j^+)(x_k^+ - x_k^-), \\ N^{\dot{\mathcal{U}}\mathcal{U}}_{\dot{\mathcal{U}}\mathcal{U}} &= (x_k^+ - x_j^+)(x_k^- - x_j^-), & N^{\dot{\mathcal{U}}\mathcal{U}}_{\mathcal{D}\mathcal{X}} &= (x_k^+ - x_j^+)(x_k^- - x_k^+), \\ N^{\mathcal{D}\mathcal{X}}_{\dot{\mathcal{U}}\mathcal{U}} &= (x_k^+ - x_j^+)(x_j^- - x_j^+), & N^{\mathcal{D}\mathcal{X}}_{\mathcal{D}\mathcal{X}} &= (x_k^+ - x_j^+)(x_k^+ - x_j^+), \end{aligned} \tag{4.15}$$

All of these expressions have the correct one-loop limit (4.9).

As a first and important test of this S-matrix we have checked the validity of three features: Parity invariance

$$S^{b_k b_j}_{a_k a_j}(x_k, x_j) = (-1)^{[a_j][a_k]+[b_j][b_k]} S^{b_j b_k}_{a_j a_k}(-x_j, -x_k) \tag{4.16}$$

and the unitarity condition

$$(-1)^{[b_j][b_k]+[c_j][c_k]} S^{b_k b_j}_{a_k a_j}(x_k, x_j) S^{c_j c_k}_{b_j b_k}(x_j, x_k) = \delta^{c_k}_{a_k} \delta^{c_j}_{a_j} \tag{4.17}$$

are rather easy to confirm. Here $[a]$ is the grading of the particle labelled by a; it is even for bosons and odd for fermions. We have verified the Yang-Baxter equation

$$\begin{aligned} &(-1)^{[b_j][a_\ell]+[b_j][b_\ell]} S^{b_k b_j}_{a_k a_j}(p_k, p_j) S^{c_k b_\ell}_{b_k a_\ell}(p_k, p_\ell) S^{c_j c_\ell}_{b_j b_\ell}(p_j, p_\ell) \\ = &(-1)^{[b_j][b_\ell]+[b_j][c_\ell]} S^{b_j b_\ell}_{a_j a_\ell}(p_j, p_\ell) S^{b_k c_\ell}_{a_k b_\ell}(p_k, p_\ell) S^{c_k c_j}_{b_k b_j}(p_k, p_j) \end{aligned} \tag{4.18}$$

in `Mathematica`. The signs arise from the application of the second S-matrix $S_{1,3}$ which requires to bring the excitations with momenta p_k and p_ℓ next to each other, i.e. we must permute p_j with p_ℓ.

[18] In our convention there are no signs if the two involved excitations for $S_{k,j}$ are adjacent, i.e. when $j = k + 1$. Otherwise various signs arise from permuting with intermediate fields.

4.3 Nested Asymptotic Bethe Ansatz for $\mathfrak{su}(1,1|2)$

For the second level of the nested Bethe ansatz we have to specify a new vacuum. As before we shall pick the scalar $\mathcal{X}$. The elementary excitations of this vacuum are given by the two fermions $\mathcal{U}$ and $\dot{\mathcal{U}}$. Although $\mathcal{DZ}$ used to be an elementary excitation of the first level $\mathcal{Z}$-vacuum, it is no longer elementary in a sea of $\mathcal{X}$'s. This can be inferred from the above S-matrix: While the fermions $\mathcal{U},\dot{\mathcal{U}}$ are stable, the derivative $\mathcal{DZ}$ is not. In other words, according to (4.13), the fermions can only flip positions with $\mathcal{X}$'s while according to (4.15) $\mathcal{DZ}$ can decay into two fermions $\mathcal{U},\dot{\mathcal{U}}$ using one of vacuum fields $\mathcal{X}$. We should therefore consider $\mathcal{DZ}$ as a double excitation in great similarity to the multiple excitations $\mathcal{D}^k\mathcal{Z}, \mathcal{D}^k\mathcal{X}, \mathcal{D}^k\mathcal{U}, \mathcal{D}^k\dot{\mathcal{U}}$ of the vacuum $\mathcal{Z}$.

To perform the nested Bethe ansatz for the system of the two fermions $\mathcal{U}$ and $\dot{\mathcal{U}}$ we can largely rely on the results of Sec. 3.4: The propagation of a single excitation and the scattering of two excitations of type $\mathcal{U}$ have been solved there. As the two flavors of fermions are very similar, the propagation and scattering among $\dot{\mathcal{U}}$'s works precisely the same way. The only point left to be investigated is the scattering between a $\mathcal{U}$ and a $\dot{\mathcal{U}}$.

Before we consider scattering, let us briefly review propagation. A single-excitation eigenstate $|x_\mathcal{U}\rangle$ with some fixed value for x is described by a wave function $\psi_k(x)$

$$|x_\mathcal{U}\rangle = \sum_{k=1}^{K_2} \psi_k(x)\, |k_\mathcal{U}\rangle, \qquad |k_\mathcal{U}\rangle = |\mathcal{X}\ldots\mathcal{X}\overset{\overset{k}{\downarrow}}{\mathcal{U}}\mathcal{X}\ldots\mathcal{X}\rangle \tag{4.19}$$

We are interested in the wave function $\psi_k(x)$ and the eigenvalue of the total scattering operator

$$s_{k,k-K_2+1}\cdots s_{k,k-1}|x_\mathcal{U}\rangle = \lambda_k|x_\mathcal{U}\rangle. \tag{4.20}$$

Note that we assume the indices k specifying the position on the reduced lattice to be periodic. The solution can be found by induction on the number of $s_{k,k-j}$'s applied to $|x_\mathcal{U}\rangle$. We shall denote the state after j steps by

$$|x_\mathcal{U}^{(k,j)}\rangle = s_{k,k-j}\cdots s_{k,k-1}|x_\mathcal{U}\rangle = \sum_{l=1}^{K_2} \psi_l^{(k,j)}(x)|l_\mathcal{U}\rangle. \tag{4.21}$$

The induction is based on the assumption

$$\frac{\psi_k^{(k,j-1)}(x)}{\psi_{k-j}^{(k,j-1)}(x)} = \frac{x - x_{2,k-j}^-}{x - x_{2,k}^+}\,. \tag{4.22}$$

which ensures that in the induction step $s_{k,k-j}|x_\mathcal{U}^{(k,j-1)}\rangle = |x_\mathcal{U}^{(k,j)}\rangle$ we can make use of the following two identities of the S-matrix

$$\begin{aligned}
s^{\mathcal{UX}}_{\mathcal{UX}}(x_k,x_j) + \frac{x-x_k^+}{x-x_j^-}\, s^{\mathcal{UX}}_{\mathcal{XU}}(x_k,x_j) &= \frac{x-x_j^+}{x-x_j^-}\,,\\
\frac{x-x_j^-}{x-x_k^+}\, s^{\mathcal{XU}}_{\mathcal{UX}}(x_k,x_j) + s^{\mathcal{XU}}_{\mathcal{XU}}(x_k,x_j) &= \frac{x_k^- - x}{x_k^+ - x}\,,
\end{aligned} \tag{4.23}$$

which hold for any x, x_k, x_j. We then find that only two elements of the wave function change

$$\psi_k^{(k,j)} = \psi_k^{(k,j-1)} \cdot \frac{x - x^+_{2,k-j}}{x - x^-_{2,k-j}}\,, \qquad \psi_{k-j}^{(k,j)} = \psi_{k-j}^{(k,j-1)} \cdot \frac{x^-_{2,k} - x}{x^+_{2,k} - x}\,. \tag{4.24}$$

It is easy to see that the induction condition of the next step is satisfied and that the wave function after j steps is given by

$$\psi_{k-m}^{(k,j)} = \psi_{k-m} \cdot \begin{cases} \prod_{l=1}^{j} \frac{x - x^+_{2,k-l}}{x - x^-_{2,k-l}} & \text{if } m = 0, \\ \frac{x^-_{2,k} - x}{x^+_{2,k} - x} & \text{if } 0 < m \le j, \\ 1 & \text{if } m > j. \end{cases} \tag{4.25}$$

Note that the assumption (4.22) for $j = 1$ immediately determines the wave-function up to an overall constant in agreement with (3.44)

$$\psi_k(x) = \psi_1(x) \prod_{j=1}^{k-1} \frac{x - x^-_{2,j}}{x - x^+_{2,j+1}}\,. \tag{4.26}$$

In particular when we impose the periodicity condition $\psi_k(x) = \psi_{k+K_2}(x)$ we find the Bethe equation (3.56)

$$\prod_{j=1}^{K_2} \frac{x - x^-_{2,j}}{x - x^+_{2,j}} = 1\,, \tag{4.27}$$

which determines the admissible values of x. It is then not difficult to see that after $j = K_2 - 1$ steps all elements of the wave function (4.25) have been multiplied by (3.45)

$$\lambda_k = \frac{x^-_{2,k} - x}{x^+_{2,k} - x} \tag{4.28}$$

and even the element k itself by virtue of (4.27).

Let us now proceed to the two-excitation problem composed from the basis states

$$\begin{aligned}
|k_1, k_3\rangle &= +|\mathcal{X}\ldots\mathcal{X}\overset{\substack{k_1\\ \downarrow}}{\mathcal{U}}\mathcal{X}\ldots\mathcal{X}\overset{\substack{k_3\\ \downarrow}}{\dot{\mathcal{U}}}\mathcal{X}\ldots\mathcal{X}\rangle \quad \text{when } k_1 < k_3, \\
|k_1, k_1\rangle &= +|\mathcal{X}\ldots\mathcal{X}\overset{\substack{k_1\\ \downarrow}}{(\mathcal{D}\mathcal{Z})}\mathcal{X}\ldots\mathcal{X}\rangle, \\
|k_1, k_3\rangle &= -|\mathcal{X}\ldots\mathcal{X}\overset{\substack{k_3\\ \downarrow}}{\dot{\mathcal{U}}}\mathcal{X}\ldots\mathcal{X}\overset{\substack{k_1\\ \downarrow}}{\mathcal{U}}\mathcal{X}\ldots\mathcal{X}\rangle \quad \text{when } k_1 > k_3,
\end{aligned} \tag{4.29}$$

We propose that the periodic scattering eigenstate is simply given by

$$|x_1, x_3\rangle = \sum_{k_1,k_3=1}^{K_2} \psi_{k_1}(x_1)\, \psi_{k_3}(x_3)\, |k_1, k_3\rangle\,. \tag{4.30}$$

As before, this can be proven by applying the partial chain of pairwise scatterings and showing that the following expression satisfies a recursion relation

$$|x_1, x_3{}^{(k,j)}\rangle = s_{k,k-j}\cdots s_{k,k-1}|x_1, x_3\rangle = \sum_{k_1,k_3=1}^{K_2} \psi_{k_1}^{(k,j)}(x_1)\,\psi_{k_3}^{(k,j)}(x_3)\,|k_1, k_3\rangle\,. \tag{4.31}$$

The recursion is based on the same identities as before, but we need to separately consider the case when $s_{k,k-j}$ acts on both $\mathcal{U}$ and $\dot{\mathcal{U}}$ at the same time or when it acts on $\mathcal{DZ}$. Luckily, this is guaranteed by the identity

$$s^{\mathcal{DX}}_{\mathcal{DX}} + \frac{x_3 - x_k^+}{x_3 - x_j^-}\,s^{\mathcal{DX}}_{\mathcal{U}\dot{\mathcal{U}}} - \frac{x_1 - x_k^+}{x_1 - x_j^-}\,s^{\mathcal{DX}}_{\dot{\mathcal{U}}\mathcal{U}} + \frac{x_1 - x_k^+}{x_1 - x_j^-}\,\frac{x_3 - x_k^+}{x_3 - x_j^-}\,s^{\mathcal{DX}}_{\mathcal{XD}} = \frac{x_1 - x_j^+}{x_1 - x_j^-}\,\frac{x_3 - x_j^+}{x_3 - x_j^-}\,. \tag{4.32}$$

and three similar ones.

As the two-particle wave function is merely the product of two one-particle wave functions, there is no phase shift and periodicity is ensured by two instances of (4.27), i.e.

$$\prod_{j=1}^{K_2} \frac{x_1 - x_{2,j}^-}{x_1 - x_{2,j}^+} = 1\,, \qquad \prod_{j=1}^{K_2} \frac{x_3 - x_{2,j}^-}{x_3 - x_{2,j}^+} = 1\,, \tag{4.33}$$

and the eigenvalue of the total scatting operator is

$$\lambda_k = \frac{x_{2,k}^- - x_1}{x_{2,k}^+ - x_1}\,\frac{x_{2,k}^- - x_3}{x_{2,k}^+ - x_3}\,. \tag{4.34}$$

In conclusion, this means that $\mathcal{U}$ and $\dot{\mathcal{U}}$ do not feel each other's presence. The two excitations completely factorize, there is no further diagonalization required. In fact, the factorization can be traced back to the S-matrix which also factorizes in two parts. Each part governs independently the behavior of one type of constituent oscillator $\mathbf{A}$ and $\dot{\mathbf{A}}$ introduced in Sec. 4.1. Also the identities of the sort (4.32) are essentially the product of two identities from (4.23).

Note that the factorization is slightly different from the one for two alike fermionic excitations which are subject to the Pauli principle $|x_{\mathcal{U}}, x_{\mathcal{U}}\rangle = 0$. Here the state $|x_1, x_3\rangle$ with $x_1 = x_3$ does exist and the exclusion principle is avoided by having two flavors of fermions at our disposal. The exclusion principle applies only to excitations of the same kind.

4.4 Asymptotic Bethe Equations for $\mathfrak{su}(1,1|2)$

We have now obtained the pairwise scattering of excitations $x_{1,2,3}$. As the S-matrix obeys the YBE, the factorized scattering of more than two excitations is self-consistent. We can therefore write down the Bethe equations from the results on scattering phases from

above. The asymptotic Bethe equations for the $\mathfrak{su}(1,1|2)$ sector read

$$
\begin{aligned}
1 &= \prod_{j=1}^{K_2} \frac{x_{1,k} - x_{2,j}^{+\eta_1}}{x_{1,k} - x_{2,j}^{-\eta_1}}\,, \\
1 &= \left(\frac{x_{2,k}^-}{x_{2,k}^+}\right)^L \prod_{\substack{j=1\\ j\neq k}}^{K_2} \left(\frac{1 - g^2/2x_{2,k}^+ x_{2,j}^-}{1 - g^2/2x_{2,k}^- x_{2,j}^+}\, \sigma^2(x_{2,k}, x_{2,j}) \right) \\
&\quad \times \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{x_{2,k}^{+\eta_1} - x_{2,j}^{-\eta_1}}{x_{2,k}^{-\eta_2} - x_{2,j}^{+\eta_2}} \prod_{j=1}^{K_1} \frac{x_{2,k}^{-\eta_1} - x_{1,j}}{x_{2,k}^{+\eta_1} - x_{1,j}} \prod_{j=1}^{K_3} \frac{x_{2,k}^{-\eta_2} - x_{3,j}}{x_{2,k}^{+\eta_2} - x_{3,j}}\,, \\
1 &= \prod_{j=1}^{K_2} \frac{x_{3,k} - x_{2,j}^{+\eta_2}}{x_{3,k} - x_{2,j}^{-\eta_2}}\,.
\end{aligned}
\tag{4.35}
$$

Here we have presented four equivalent forms labelled by the grading constants $\eta_1 = \pm 1$, $\eta_2 = \pm 1$ related to four different Dynkin diagrams and representation vectors. In the nested Bethe ansatz, these come about by choosing, at the second level, one of the four primary excitations $\mathcal{X}, \mathcal{U}, \dot{\mathcal{U}}, \mathcal{DZ}$ as the new vacuum. Above we have restricted ourselves to $\mathcal{X}$ which corresponds to $\eta_1 = \eta_2 = +1$. In Sec. 4.7 we shall show how to dualize between the different forms and thus show their equivalence independently of the nested Bethe ansatz.

As above, $\sigma(x_k, x_j)$ determines the model: We set $\sigma(x_k, x_j) = 1$ for gauge theory and $\sigma(x_k, x_j) \neq 1$ as in (2.46) for the string chain. The local charges are obtained as before and determined only through the middle, momentum-carrying roots

$$
Q_r = \sum_{j=1}^{K_2} q_r(x_{2,j}) \tag{4.36}
$$

with $q_r(x)$ given in (2.31). The anomalous dimension is given by (2.41) with $K = K_2$ and for (cyclic) gauge theory states, the momentum constraint (2.40) must be obeyed. Note that for $K_1 = K_3 = 0$, the equations reduce to either one of the three rank-one sectors in (2.39,2.48) specified by $\eta = (\eta_1 + \eta_2)/2$.

The Dynkin labels $[q_1, p, q_2]$ of $\mathfrak{su}(4)$ are related to the excitation numbers by

$$
\begin{aligned}
q_1 &= -\eta_1 K_1 + \tfrac{1}{2}(1+\eta_1) K_2, \\
p &= +L + \eta_1 K_1 - \tfrac{1}{2}(2 + \eta_1 + \eta_2) K_2 + \eta_2 K_3, \\
q_2 &= -\eta_2 K_3 + \tfrac{1}{2}(1+\eta_2) K_2
\end{aligned}
\tag{4.37}
$$

while the labels $[s_1, r, s_2]$ of $\mathfrak{su}(2,2)$ are given by

$$
\begin{aligned}
s_1 &= +\eta_1 K_1 + \tfrac{1}{2}(1-\eta_1) K_2, \\
r &= -L - \eta_1 K_1 - \tfrac{1}{2}(2 - \eta_1 - \eta_2) K_2 - \eta_2 K_3 - \delta D, \\
s_2 &= +\eta_2 K_3 + \tfrac{1}{2}(1-\eta_2) K_2.
\end{aligned}
\tag{4.38}
$$

4.5 Multiplet Splitting

The $\mathfrak{su}(1,1|2)$ sector has a hidden $\mathfrak{psu}(1|1) \times \mathfrak{psu}(1|1)$ symmetry [25]. Furthermore, all these factors have a common central charge, the anomalous dimension δD, and they share two abelian external automorphisms, the length L and the hypercharge B. The symmetry factors originate from the full $\mathfrak{psu}(2,2|4)$ symmetry. The $\mathfrak{psu}(1|1)$ algebras are hidden symmetries because they act trivially at leading order, i.e. at $g = 0$.

There are two types of multiplets of $\mathfrak{psu}(1|1) \ltimes \mathfrak{u}(1)$, a singlet $\mathbf{1}$ and a doublet $\mathbf{1}|\mathbf{1}$. The singlet has zero central charge, in the fully interacting theory it is realized only by the vacuum. The other states have a non-zero central charge δD and therefore come in doublets. However, at the classical level, the anomalous dimension vanishes and the doublets split into two singlets.

Let us see how the Bethe ansatz realizes the hidden $\mathfrak{psu}(1|1)$ symmetry. For the one-loop Bethe ansatz, the multiplet splitting implies that each state should have a partner with precisely the same $\mathfrak{su}(1,1|2)$ quantum numbers, energy and local charges. Moreover these two multiplets should join in the higher-loop Bethe ansatz.

Consider a spin chain eigenstate of length L with (K_1, K_2, K_3) excitations. Assume that one of the x_1-roots is at $x = 0$, i.e. $x_{1,K_1} = 0$. The corresponding Bethe equation (4.35) reads

$$1 = \prod_{j=1}^{K_2} \frac{x_{2,j}^+}{x_{2,j}^-}\,. \tag{4.39}$$

Curiously, this is just the cyclicity constraint (2.40). In other words, $x = 0$ can be a x_1-root if and only if the momentum constraint is satisfied. Let us see what the effects of $x_{1,K_1} = 0$ on the other roots are. After substitution, the middle equation of (4.35) turns out to be precisely the equation for a spin chain of length $L + \eta_1$ with $(K_1 - 1, K_2, K_3)$ excitations. The outer two equations are not modified as there are no self-interactions of x_1's and no interactions between x_1's and x_3's.

We conclude that for every eigenstate with a root $x_1 = 0$, there exist an eigenstate with $L' = L + \eta_1$ and $(K_1', K_2', K_3') = (K_1 - 1, K_2, K_3)$ with the same set of Bethe roots (except for $x_1 = 0$). Conversely, for every state in the zero-momentum sector without the root $x_1 = 0$ there exist a state with $L' = L - \eta_1$ and $x_1 = 0$ as the only additional root. As the energy and local charges are determined through the x_2's alone, they coincide for both states. They consequently form a doublet of $\mathfrak{psu}(1|1)$. A similar construction applies for roots $x_3 = 0$ which explains the appearance of the second $\mathfrak{psu}(1|1)$ factor.

Note that an essential requirement of this construction is the restriction to cyclic states. The hidden $\mathfrak{psu}(1|1)$ symmetry in fact does not exist for generic states. This parallels the observations in the construction of a similar spin chain in [25], but now at the level of the Bethe equations. It does not mean, however, that the spin chain, Bethe ansatz or $\mathfrak{su}(1,1|2)$ symmetry should be restricted to cyclic states, they are valid for states with arbitrary total momentum as well.

We can also explain this mechanism at a somewhat deeper level of the integrable structure. In the standard Bethe ansätze, multiplets are realized as follows: The highest-weight state has no Bethe root at $u = \infty$. Descendants are obtained from the primary state by adding roots at $u = \infty$. These neither change the Bethe equations, nor do they

modify the energy or higher charge eigenvalues. In fact, the point $u = \infty$ is related to the generators of the symmetry algebra.

The situation for the higher-loop Bethe ansätze is somewhat different. Our spectral parameter x resembles the one of string sigma models, see [9, 29]. There, not only the point $x = \infty$ relates to the symmetry generators, but, not surprisingly, also the point $x = 0$. In the classical limit, $g = 0$, the x-plane limits to the u-plane. The points at infinity are identified, but the limit at the point zero is singular [11, 12]. This feature spoils the relation between $u = 0$ and the symmetry generators, whereas it is apparent for $x = 0$. This explains the multiplet splitting/joining mechanism in the Bethe ansatz.

4.6 Exact Degeneracies

There is an even larger set of hidden symmetries that can boost the degeneracy of states by several factors of two. Let us for simplicity set $\eta_1 = \eta_2 = +1$. Consider an eigenstate which has a Bethe root x of flavor 1 which is *not* also a root of flavor 3. Then we can construct a state with the same set of Bethe roots, but now x being of flavor 3 instead of 1. This state obeys the Bethe equations (4.35) because the first and third equation coincide and the second equation does not distinguish between x_1 and x_3. The states have the same higher charges, because they share all the roots of flavor 2. Nevertheless, the representation of both states is different due to different occupation numbers for the individual flavors.

Degenerate states within the same representation can also be constructed. Suppose there are at least two roots to which the above argument applies. For a pair of roots, we can associate one to flavor 1 and the other to flavor 3 or vice versa. In both cases the excitation numbers are the same and thus we get two completely degenerate multiplets of states. These even have the same parity and therefore this effect is not related to the pairing described in [4].

We see that (except for the global charges) we do not have to distinguish between roots of type 1 and 3. Nevertheless, one has to keep in mind that no two Bethe roots can occupy the same position. For these mixed roots the exclusion principle is circumvented and we can have two roots at the same position, c.f. Sec. 4.3, they merely have to be associated to different flavors. The situation is somewhat reminiscent of $\mathcal{N} = 2$ supersymmetry with two flavors of fermions.

It would be interesting to investigate this degeneracy further. What is its origin and is it restricted to planar gauge theory only? Are these degeneracies restricted to the $\mathfrak{su}(1,1|2)$ sector or are there even larger degeneracies in the full theory which are not directly related to the symmetry group?

4.7 Duality Transformation

Here we shall show the equivalence of the Bethe equations for different gradings $\eta_1 = \pm 1$, $\eta_2 = \pm 1$. This is an essential step in demonstrating the self-consistency of the Bethe ansatz. The proof parallels the one for the one-loop level in [49, 55, 14].

We rewrite the Bethe equation for fermionic roots of type x_1 as the algebraic equation

$P(x_{1,k}) = 0$ with the polynomial

$$P(x) = \prod_{j=1}^{K_2} \bigl(x - x_{2,j}^{+}\bigr) - \prod_{j=1}^{K_2} \bigl(x - x_{2,j}^{-}\bigr). \tag{4.40}$$

Clearly, all the roots $x_{1,k}$ solve the same equation, but there are $\tilde{K}_1 = K_2 - K_1 - 1$ further solutions $\tilde{x}_{1,k}$. We can thus write the polynomial as the product of monomials

$$P(x) \sim \prod_{j=1}^{K_1} (x - x_{1,j}) \prod_{j=1}^{\tilde{K}_1} (x - \tilde{x}_{1,j}) \tag{4.41}$$

with some fixed factor of proportionality. Using this identity, we can now write some combination of terms which appear in the Bethe equations (4.35) using the polynomial

$$\prod_{j=1}^{K_1} \frac{x_{2,k}^{+\eta_1} - x_{1,j}}{x_{2,k}^{-\eta_1} - x_{1,j}} \prod_{j=1}^{\tilde{K}_1} \frac{x_{2,k}^{+\eta_1} - \tilde{x}_{1,j}}{x_{2,k}^{-\eta_1} - \tilde{x}_{1,j}} = \frac{P(x_{2,k}^{+\eta_1})}{P(x_{2,k}^{-\eta_1})}\,. \tag{4.42}$$

When we substitute the original definition of the polynomials (4.40) we obtain a different expression

$$\frac{P(x_{2,k}^{+\eta_1})}{P(x_{2,k}^{-\eta_1})} = \frac{\prod_{j=1}^{K_2} \bigl(x_{2,k}^{+\eta_1} - x_{2,j}^{+}\bigr) - \prod_{j=1}^{K_2} \bigl(x_{2,k}^{+\eta_1} - x_{2,j}^{-}\bigr)}{\prod_{j=1}^{K_2} \bigl(x_{2,k}^{-\eta_1} - x_{2,j}^{+}\bigr) - \prod_{j=1}^{K_2} \bigl(x_{2,k}^{-\eta_1} - x_{2,j}^{-}\bigr)} = \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{x_{2,k}^{+\eta_1} - x_{2,j}^{-\eta_1}}{x_{2,k}^{-\eta_1} - x_{2,j}^{+\eta_1}}\,. \tag{4.43}$$

For $\eta_1 = +1$, the first term in the numerator and the second term in the denominator are trivially zero and vice versa for $\eta_1 = -1$. The sign is cancelled by removing the factor with $j = k$ which is always -1. Combining the two equations we find an identity which relates terms of the Bethe equations using the roots x_1 and their duals $\tilde{x}_1$

$$\prod_{\substack{j=1\\ j\neq k}}^{K_2} \bigl(x_{2,k}^{+\eta_1} - x_{2,j}^{-\eta_1}\bigr) \prod_{j=1}^{K_1} \frac{x_{2,k}^{-\eta_1} - x_{1,j}}{x_{2,k}^{+\eta_1} - x_{1,j}} = \prod_{\substack{j=1\\ j\neq k}}^{K_2} \bigl(x_{2,k}^{-\eta_1} - x_{2,j}^{+\eta_1}\bigr) \prod_{j=1}^{\tilde{K}_1} \frac{x_{2,k}^{+\eta_1} - \tilde{x}_{1,j}}{x_{2,k}^{-\eta_1} - \tilde{x}_{1,j}}\,. \tag{4.44}$$

We apply this identity to the middle equation of (4.35) and invert the first equation to obtain the dualized equations

$$\begin{aligned}
1 &= \prod_{j=1}^{K_2} \frac{\tilde{x}_{1,k} - x_{2,j}^{-\eta_1}}{\tilde{x}_{1,k} - x_{2,j}^{+\eta_1}}\,,\\
1 &= \left(\frac{x_{2,k}^{-}}{x_{2,k}^{+}}\right)^{L} \prod_{\substack{j=1\\ j\neq k}}^{K_2} S_0(x_{2,k}, x_{2,j}) \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{x_{2,k}^{-\eta_1} - x_{2,j}^{+\eta_1}}{x_{2,k}^{-\eta_2} - x_{2,j}^{+\eta_2}} \prod_{j=1}^{\tilde{K}_1} \frac{x_{2,k}^{+\eta_1} - \tilde{x}_{1,j}}{x_{2,k}^{-\eta_1} - \tilde{x}_{1,j}} \prod_{j=1}^{K_3} \frac{x_{2,k}^{-\eta_2} - x_{3,j}}{x_{2,k}^{+\eta_2} - x_{3,j}}\,,\\
1 &= \prod_{j=1}^{K_2} \frac{x_{3,k} - x_{2,j}^{+\eta_2}}{x_{3,k} - x_{2,j}^{-\eta_2}}\,.
\end{aligned} \tag{4.45}$$

They agree with the original equations after substituting $\eta_1 \mapsto -\eta_1$ and $x_{1,k} \to \tilde{x}_{1,k}$. As the x_2-roots remain unchanged, the energy and local charges remain invariant under dualization. For $K_3 = 0$, this proves the equivalence of the two sets of equations in Sec. 3.4 on an independent basis. The argument for roots of type 3 is the same.

Note that we will not try to dualize the middle node of the Dynkin diagram. This would take the protected ground state of scalar fields $\mathcal{Z}$ into a highly interacting pseudo-vacuum of fermions $\mathcal{U}$. Consequently, and in contrast to the above transformation as well as the one-loop approximation, the dualization of the middle node appears to be substantially more involved.

4.8 Frolov-Tseytlin Limit

Here we present the Frolov-Tseytlin limit of the Bethe equations. In this limit, the duality transformation is a mere permutation Riemann sheets [14], so without loss of generality we can set $\eta_1 = \eta_2 = +1$. The limit of the Bethe equations for gauge theory with $\sigma(x_k, x_j) = 1$ reads[19]

$$\begin{aligned}
-2\pi n_{1,a} &= -G_2(x) && \text{for } x \in \mathcal{C}_{1,a}, \\
-2\pi n_{2,a} &= +2\not{H}_2(x) + \frac{L/x}{1-g^2/2x^2} \\
&\quad - G_1(x) - G_1'(0)\,\frac{g^2/2x}{1-g^2/2x^2} - G_1(0)\,\frac{g^2/2x^2}{1-g^2/2x^2} \\
&\quad - G_3(x) - G_3'(0)\,\frac{g^2/2x}{1-g^2/2x^2} - G_3(0)\,\frac{g^2/2x^2}{1-g^2/2x^2} && \text{for } x \in \mathcal{C}_{2,a}, \\
-2\pi n_{3,a} &= -G_2(x) && \text{for } x \in \mathcal{C}_{3,a}.
\end{aligned} \tag{4.46}$$

The limits of the equations for the string chain with $\sigma(x_k, x_j) \neq 1$ are

$$\begin{aligned}
-2\pi n_{1,a} &= -G_2(x) && \text{for } x \in \mathcal{C}_{1,a}, \\
-2\pi n_{2,a} &= +2\not{G}_2(x) + 2G_2'(0)\,\frac{g^2/2x}{1-g^2/2x^2} + \frac{L/x}{1-g^2/2x^2} \\
&\quad - G_1(x) - G_1'(0)\,\frac{g^2/2x}{1-g^2/2x^2} - G_1(0)\,\frac{g^2/2x^2}{1-g^2/2x^2} \\
&\quad - G_3(x) - G_3'(0)\,\frac{g^2/2x}{1-g^2/2x^2} - G_3(0)\,\frac{g^2/2x^2}{1-g^2/2x^2} && \text{for } x \in \mathcal{C}_{2,a}, \\
-2\pi n_{3,a} &= -G_2(x) && \text{for } x \in \mathcal{C}_{3,a}.
\end{aligned} \tag{4.47}$$

They perfectly agree with the expressions derived from the classical superstring sigma model in [12]. We have thus found a possible quantization for superstrings on $AdS_5 \times S^5$ restricted to the subspace $AdS_3 \times S^3$ in the spirit of [15, 16].

[19] For many states, Bethe roots form strings of stacks, not merely strings of roots [14]. The corresponding cuts stretch between several resolvents G_j. To handle this situation carefully, additional resolvents and equations should be introduced. Nevertheless, the presented equations contain all the relevant information and it is not necessary to specify the extended equations.

5 Complete Asymptotic Bethe Equations

The $\mathfrak{su}(1,1|2)$ sector is the largest sector for which mixing of states of different lengths is suppressed at all orders in perturbation theory. Nevertheless, the dynamic nature of the higher-loop spin chain for $\mathcal{N}=4$ gauge theory is not an obvious obstacle for integrability. Indeed some signs of higher-loop integrability beyond $\mathfrak{su}(1,1|2)$ were found in [27]. We might therefore hope that the spectrum of the full model with $\mathfrak{psu}(2,2|4)$ symmetry can also be described by a suitable Bethe ansatz. In this section we will assemble various pieces of the puzzle available in the literature and construct candidate Bethe equations for the complete higher-loop spin chain of $\mathcal{N}=4$ SYM and the complete string chain.

5.1 Bethe Equations

Before we make a proposal for the equations, let us present a list of constraints and expected features (see also [25]):

i. The higher-loop equations should turn into the one-loop equations of [3] when setting $g=0$.

ii. The equations should turn into the equations of the previous section when restricting to the $\mathfrak{su}(1,1|2)$ sector.

iii. The thermodynamic limit of the string chain equations ($\sigma \neq 1$ as in (2.46)) should agree with [12].

iv. The length L and the hypercharge B are not conserved quantities at higher loops. This fact should be reflected by the equations.

v. In the Bethe ansatz, the highest-weight state of a multiplet is singled out by the absence of Bethe roots at $u=\infty$. In the x-plane this point corresponds to $x=\infty$ or $x=0$. Bethe roots at $u=\infty$ should indicate descendants.

vi. All short multiplets in the free theory (except the vacuum) must join into long multiplets in the interacting theory. For the Bethe ansatz this implies that some states should appear as primaries at $g=0$, but become descendants when $g\neq 0$.

vii. The spin chains from gauge theory are defined only modulo cyclic permutations. While at one loop this feature merely led to the restriction to the zero-momentum sector of a general periodic spin chain, the higher-loop spin chain apparently is self-consistent only in the zero-momentum sector [27, 25]. The two main reasons are: Firstly, length-changing interactions destroy the identification of individual sites and allow only for relative positions. Secondly, multiplet joining cannot work due to a mismatch of states (unless there are many more protected states than expected).

viii. One should be able to read the Dynkin labels of a state from the set of Bethe roots: When expanding the left hand side of the Bethe equation for a root of flavor j around $x=\infty$ while keeping all other roots fixed, one should obtain the j-th Dynkin label r_j via $1 - ir_j/x_{j,k} + \mathcal{O}(1/x_{j,k}^2)$, see, e.g., (2.60). In particular, for

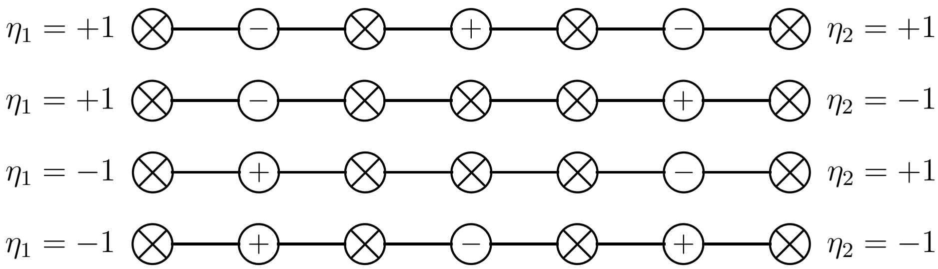

Figure 1: Dynkin diagrams of $\mathfrak{su}(2,2|4)$ for the gradings $\eta_1, \eta_2 = \pm 1$. The signs in the white nodes indicate the sign of the diagonal elements of the Cartan matrix.

the non-compact algebra $\mathfrak{psu}(2,2|4)$ some Dynkin labels contain the anomalous dimension δD.

ix. Somehow the very nature of the algebra $\mathfrak{psu}(2,2|4)$ should play a role because the $\mathcal{N} = 4$ superconformal field theory is a very special model. Unlike the standard spin chains which can be constructed for an arbitrary symmetry algebra, the higher-loop spin chain is expected to make use of special features of $\mathfrak{psu}(2,2|4)$.

x. There should be several equivalent formulations of the Bethe equations for various forms of the Dynkin diagram of $\mathfrak{psu}(2,2|4)$.

xi. The spectrum should agree with $\mathcal{N} = 4$ SYM.

We have found a set of Bethe equations which fulfills all of the above conditions or at least does not apparently violate them (as for point *xi*). There are four forms labelled by the gradings $\eta_1, \eta_2 = \pm 1$. They correspond to different choices for the Cartan matrix of $\mathfrak{su}(2,2|4)$

$$M_{j,j'} = \begin{pmatrix} & +\eta_1 & & & & & \\ +\eta_1 & -2\eta_1 & +\eta_1 & & & & \\ & +\eta_1 & & -\eta_1 & & & \\ & & -\eta_1 & +\eta_1+\eta_2 & -\eta_2 & & \\ & & & -\eta_2 & & +\eta_2 & \\ & & & & +\eta_2 & -2\eta_2 & +\eta_2 \\ & & & & & +\eta_2 & \end{pmatrix} \tag{5.1}$$

given by the Dynkin diagrams in Fig. 1. The Bethe equations are presented in Tab. 5.

Point *ii* is easily confirmed by setting $K_1 = K_2 = K_6 = K_7 = 0$ to restrict to the $\mathfrak{su}(1,1|2)$ sector. The remaining equations agree with (4.35). Point *i* is almost as straight-forward: One has to set $x = u$, $x^\pm = u \pm \frac{i}{2}$ with a finite u and $g = 0$.[20] Then we compare to the Bethe equations in [3] using the appropriate Dynkin diagram in Fig. 1. Concerning point *v* we note that adding a root of any kind at $x = \infty$ does not modify the original equations or the set of higher charge eigenvalues. Thus a state with roots at $x = \infty$ is a descendant and a state without such roots is highest weight.

[20] The limit is subtle: For the naive limiting equations we assume that all $u = \mathcal{O}(g^0)$. Special precautions have to be taken when some $u = \mathcal{O}(g^1)$ (singular states) or $u = \mathcal{O}(g^2)$ (dynamic transformation), see below.

$$1 = \prod_{j=1}^{K_4} \frac{x_{4,j}^+}{x_{4,j}^-} \,,$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + \frac{i}{2}\eta_1}{u_{1,k} - u_{2,j} - \frac{i}{2}\eta_1} \prod_{j=1}^{K_4} \frac{1 - g^2/2x_{1,k}x_{4,j}^{+\eta_1}}{1 - g^2/2x_{1,k}x_{4,j}^{-\eta_1}} \,,$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{u_{2,k} - u_{2,j} - i\eta_1}{u_{2,k} - u_{2,j} + i\eta_1} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} + \frac{i}{2}\eta_1}{u_{2,k} - u_{3,j} - \frac{i}{2}\eta_1} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + \frac{i}{2}\eta_1}{u_{2,k} - u_{1,j} - \frac{i}{2}\eta_1} \,,$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}\eta_1}{u_{3,k} - u_{2,j} - \frac{i}{2}\eta_1} \prod_{j=1}^{K_4} \frac{x_{3,k} - x_{4,j}^{+\eta_1}}{x_{3,k} - x_{4,j}^{-\eta_1}} \,,$$

$$1 = \left(\frac{x_{4,k}^-}{x_{4,k}^+}\right)^L \prod_{\substack{j=1\\ j\neq k}}^{K_4} \left(\frac{x_{4,k}^{+\eta_1} - x_{4,j}^{-\eta_1}}{x_{4,k}^{-\eta_2} - x_{4,j}^{+\eta_2}} \, \frac{1 - g^2/2x_{4,k}^+ x_{4,j}^-}{1 - g^2/2x_{4,k}^- x_{4,j}^+} \, \sigma^2(x_{4,k}, x_{4,j}) \right)$$

$$\times \prod_{j=1}^{K_1} \frac{1 - g^2/2x_{4,k}^{-\eta_1} x_{1,j}}{1 - g^2/2x_{4,k}^{+\eta_1} x_{1,j}} \prod_{j=1}^{K_3} \frac{x_{4,k}^{-\eta_1} - x_{3,j}}{x_{4,k}^{+\eta_1} - x_{3,j}} \prod_{j=1}^{K_5} \frac{x_{4,k}^{-\eta_2} - x_{5,j}}{x_{4,k}^{+\eta_2} - x_{5,j}} \prod_{j=1}^{K_7} \frac{1 - g^2/2x_{4,k}^{-\eta_2} x_{7,j}}{1 - g^2/2x_{4,k}^{+\eta_2} x_{7,j}} \,,$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k} - u_{6,j} + \frac{i}{2}\eta_2}{u_{5,k} - u_{6,j} - \frac{i}{2}\eta_2} \prod_{j=1}^{K_4} \frac{x_{5,k} - x_{4,j}^{+\eta_2}}{x_{5,k} - x_{4,j}^{-\eta_2}} \,,$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_6} \frac{u_{6,k} - u_{6,j} - i\eta_2}{u_{6,k} - u_{6,j} + i\eta_2} \prod_{j=1}^{K_5} \frac{u_{6,k} - u_{5,j} + \frac{i}{2}\eta_2}{u_{6,k} - u_{5,j} - \frac{i}{2}\eta_2} \prod_{j=1}^{K_7} \frac{u_{6,k} - u_{7,j} + \frac{i}{2}\eta_2}{u_{6,k} - u_{7,j} - \frac{i}{2}\eta_2} \,,$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{7,k} - u_{6,j} + \frac{i}{2}\eta_2}{u_{7,k} - u_{6,j} - \frac{i}{2}\eta_2} \prod_{j=1}^{K_4} \frac{1 - g^2/2x_{7,k}x_{4,j}^{+\eta_2}}{1 - g^2/2x_{7,k}x_{4,j}^{-\eta_2}} \,,$$

$$Q_r = \frac{1}{r-1} \sum_{j=1}^{K_4} \left(\frac{i}{(x_{4,j}^+)^{r-1}} - \frac{i}{(x_{4,j}^-)^{r-1}} \right), \qquad \delta D = g^2 Q_2 = g^2 \sum_{j=1}^{K_4} \left(\frac{i}{x_{4,j}^+} - \frac{i}{x_{4,j}^-} \right).$$

Table 5: Asymptotic Bethe equations for the complete models. The first line is the momentum constraint which is an essential part of the Bethe ansatz. The following seven equations must hold for all $k = 1, \ldots, K_j$. The variables u and $x^\pm$ are related to the Bethe roots x by (2.30,2.28). The last line determines the local charge eigenvalues Q_r and the anomalous dimension δD. The function σ specifies the model: $\sigma = 1$ for gauge theory or as in (2.46) for the string chain. The gradings $\eta_1, \eta_2 = \pm 1$ select one of the four Dynkin diagrams in Fig. 1.

5.2 Global Charges

The Dynkin labels of a state can be read off from the Bethe equations as explained at point *viii*. We expand the left hand side of the Bethe equation for a root of flavor j around $x = \infty$ and keep all other roots fixed. We then obtain the Dynkin labels r_j as the leading coefficient[21]

$$\begin{aligned}
r_1 &= -\eta_1 K_2 - \tfrac{1}{2}\eta_1 \delta D, \\
r_2 &= -\eta_1 K_3 + 2\eta_1 K_2 - \eta_1 K_1, \\
r_3 &= +\eta_1 K_4 - \eta_1 K_2 + \tfrac{1}{2}\eta_1 \delta D, \\
r_4 &= +L - (\eta_1 + \eta_2) K_4 + \eta_1 K_3 + \eta_2 K_5 + \tfrac{1}{4}(2 - \eta_1 - \eta_2)\delta D, \\
r_5 &= +\eta_2 K_4 - \eta_2 K_6 + \tfrac{1}{2}\eta_2 \delta D, \\
r_6 &= -\eta_2 K_5 + 2\eta_2 K_6 - \eta_2 K_7, \\
r_7 &= -\eta_2 K_6 - \tfrac{1}{2}\eta_2 \delta D.
\end{aligned} \tag{5.2}$$

These Dynkin labels depend on the particular choice of Dynkin diagram. It is more convenient to use the Dynkin labels of the bosonic subalgebras. The Dynkin labels $[q_1, p, q_2]$ of $\mathfrak{su}(4)$ are given by

$$\begin{aligned}
q_1 &= -\eta_1 K_1 - (1 - \eta_1) K_2 - \eta_1 K_3 + \tfrac{1}{2}(1 + \eta_1) K_4, \\
p &= +L + \tfrac{1}{2}(1 - \eta_1) K_2 + \eta_1 K_3 - \tfrac{1}{2}(2 + \eta_1 + \eta_2) K_4 + \eta_2 K_5 + \tfrac{1}{2}(1 - \eta_2) K_6, \\
q_2 &= -\eta_2 K_7 - (1 - \eta_2) K_6 - \eta_2 K_5 + \tfrac{1}{2}(1 + \eta_2) K_4
\end{aligned} \tag{5.3}$$

and the labels $[s_1, r, s_2]$ of $\mathfrak{su}(2,2)$ read

$$\begin{aligned}
s_1 &= +\eta_1 K_1 - (1 + \eta_1) K_2 + \eta_1 K_3 + \tfrac{1}{2}(1 - \eta_1) K_4, \\
r &= -L + \tfrac{1}{2}(1 + \eta_1) K_2 - \eta_1 K_3 - \tfrac{1}{2}(2 - \eta_1 - \eta_2) K_4 - \eta_2 K_5 + \tfrac{1}{2}(1 + \eta_2) K_6 - \delta D, \\
s_2 &= +\eta_2 K_7 - (1 + \eta_2) K_6 + \eta_2 K_5 + \tfrac{1}{2}(1 - \eta_2) K_4.
\end{aligned} \tag{5.4}$$

Note that it is sufficient to state just these six charges, the seventh is determined through the central charge constraint of $\mathfrak{psu}(2,2|4)$ which reads for the present Cartan matrix

$$\eta_1 r_1 - \eta_1 r_3 + \eta_2 r_5 - \eta_2 r_7 = 0. \tag{5.5}$$

Often it is useful to know the scaling dimension $D = -\frac{1}{2}s_1 - r - \frac{1}{2}s_2$, the corresponding charge $J = \frac{1}{2}q_1 + p + \frac{1}{2}q_2$ of $\mathfrak{su}(4)$ or the plane-wave light-cone energy $D - J$

$$\begin{aligned}
J &= L + \tfrac{1}{2}\eta_1 (K_3 - K_1) - \tfrac{1}{4}(2 + \eta_1 + \eta_2) K_4 + \tfrac{1}{2}\eta_2 (K_5 - K_7), \\
D &= L + \tfrac{1}{2}\eta_1 (K_3 - K_1) + \tfrac{1}{4}(2 - \eta_1 - \eta_2) K_4 + \tfrac{1}{2}\eta_2 (K_5 - K_7) + \delta D, \\
D - J &= K_4 + \delta D.
\end{aligned} \tag{5.6}$$

[21] As expected from symmetry arguments, the result agrees for both models, i.e. for gauge and string theory. Together with the correct scaling behavior of both models, it implies the exact agreement of spectra up to two loops, cf. Sec. 2.8.

L	K_1	K_2	$(E_0, E_2, E_{4\mathrm{g}} \vert E_{4\mathrm{s}})^P$
6	4	2	$(14E-36, -24E+90, \frac{173}{2}E-315 \vert \frac{151}{2}E-279)^+ \surd$
7	4	2	$(22E^2-144E+248, -37E^2+460E-1016, 125E^2-1893E+4438 \vert 106E^2-1659E+3942)^- \surd$
8	4	2	$(7, -\frac{19}{2}, \frac{59}{2} \vert \frac{115}{4})^\pm \surd$ $(44E^5-768E^4+6752E^3-31168E^2+70528E-60224,$ $-73E^5+2486E^4-31804E^3+188280E^2-506048E+487104,$ $251E^5-10452E^4+156202E^3-1041992E^2+3055168E-3125328 \vert$ $223E^5-9500E^4+144054E^3-970456E^2+2864848E-2944656)^+ \surd$

Table 6: Spectrum of lowest-lying 4, 2-excitation states of the $\mathfrak{su}(3)$ and $\mathfrak{su}(2|1)$ sectors.

L	K_1	K_2	$(E_0, E_2, E_{4\mathrm{g}} \vert E_{4\mathrm{s}})^P$
8	5	2	$(9, -\frac{31}{2}, \frac{103}{2} \vert \frac{177}{4})^\pm$ $(24E^2-172E+344, -39E^2+524E-1372, 138E^2-2209E+6198 \vert 127E^2-2069E+5854)^- \surd$

Table 7: Spectrum of lowest-lying states genuinely in the $\mathfrak{su}(3)$ sector.

Conversely, the numbers of Bethe roots in terms of Dynkin labels of $\mathfrak{su}(4)$ and $\mathfrak{su}(2,2)$ are[22]

$$\begin{aligned}
K_1 &= +\tfrac{1}{2}\eta_1(L-B) - \tfrac{1}{8}(1-\eta_1)(2r_0+3s_1+s_2) - \tfrac{1}{8}(1+\eta_1)(2p+3q_1+q_2),\\
K_2 &= -\tfrac{1}{4}(2r_0+3s_1+s_2) - \tfrac{1}{4}(2p+3q_1+q_2),\\
K_3 &= -\tfrac{1}{2}\eta_1(L-B) - \tfrac{1}{8}(3+\eta_1)(2r_0-s_1+s_2) - s_1 - \tfrac{1}{8}(3-\eta_1)(2p-q_1+q_2) - q_1,\\
K_4 &= -r_0 - \tfrac{1}{2}s_1 - \tfrac{1}{2}s_2 - p - \tfrac{1}{2}q_1 - \tfrac{1}{2}q_2,\\
K_5 &= -\tfrac{1}{2}\eta_2(L+B) - \tfrac{1}{8}(3+\eta_2)(2r_0+s_1-s_2) - s_2 - \tfrac{1}{8}(3-\eta_2)(2p+q_1-q_2) - q_2,\\
K_6 &= -\tfrac{1}{4}(2r_0+s_1+3s_2) - \tfrac{1}{4}(2p+q_1+3q_2),\\
K_7 &= +\tfrac{1}{2}\eta_2(L+B) - \tfrac{1}{8}(1-\eta_2)(2r_0+s_1+3s_2) - \tfrac{1}{8}(1+\eta_2)(2p+q_1+3q_2)
\end{aligned} \tag{5.7}$$

with $r_0 = r + \delta D$. The label B is the $\mathfrak{u}(1)$ hypercharge of $\mathfrak{pu}(2,2|4)$.

5.3 Spectrum

Clearly, the most important constraint on the Bethe equations is *xi*, the equations must give the right answer, i.e. the predicted spectrum must actually agree with $\mathcal{N}=4$ SYM! While we are still far from proving this point, we can nevertheless compare the energies of several states with explicit computations using the three-loop $\mathfrak{su}(2|3)$ dynamic spin chain Hamiltonian of [27], i.e. up to third order in g^2.

We have computed the energies of all states for the $\mathfrak{su}(2|3)$ spin chain of length $L \leq 8$ by direct diagonalization of this Hamiltonian. For those multiplets which are part of the $\mathfrak{su}(1|2)$ sector, the results have already been presented in Tab. 1,2,3,4. The multiplets beyond the $\mathfrak{su}(1|2)$ sector can be grouped into three classes and are given in Tab. 6,7,8. The states which appear up to $L=8$ actually do not saturate the full $\mathfrak{su}(2|3)$ model, but

[22]Note that the definition of the highest-weight state of a multiplet depends on the Dynkin diagram, i.e. on η_1, η_2.

L	K_1	K_2	$(E_0, E_2, E_{4\mathrm{g}} \vert E_{4\mathrm{s}})^P$
7	5	2	$(9, -15, 51 \vert \frac{183}{4})^{\pm}$ $\surd$
8	5	2	$(8, -13, \frac{173}{4} \vert \frac{157}{4})^{\pm}$
			$(10, -\frac{67}{4}, \frac{3725}{64} \vert \frac{3437}{64})^{\pm}$
			$(19E - 86, -\frac{133}{4}E + \frac{1169}{4}, \frac{7395}{64}E - \frac{79503}{64} \vert \frac{6707}{64}E - \frac{73359}{64})^{\pm}$
8	6	2	$\big(30E^2-280E+808, -52E^2+926E-3800, \frac{361}{2}E^2-3878E+18246 \big\vert \frac{315}{2}E^2-3476E+16630\big)^{-}$
8	6	3	$\big(32E^2-324E+1032, -54E^2+1038E-4636, * \big\vert *\big)^{+}$ $\surd$

Table 8: Spectrum of lowest-lying states genuinely in the $\mathfrak{su}(2|1)$ sector.

(at one loop) they are all part of the $\mathfrak{su}(3)$ or $\mathfrak{su}(2|1)$ subsectors. We have thus given only the excitation numbers K'_1, K'_2 for the appropriate subsectors. They are related to the excitation numbers for the complete model by $K_1 = K_5 = K_6 = K_7 = 0$ and $K_4 = K'_1$, $K_3 = K'_1 - 2$, $K'_2 = K'_2 - 1$ when we specialize to $\eta_2 = -1$ for $\mathfrak{su}(3)$ and to $\eta_2 = +1$ for $\mathfrak{su}(1|2)$.

We have investigated several states from the tables using the proposed Bethe equations of Tab. 5; they are marked by "$\surd$". The agreement of the Bethe equations with the tables is perfect. This includes one paired state and one state with a relatively large number of excitations. There is one point worth mentioning concerning the state with $(K_1, K_2) = (6, 3)$ from Tab. 8 when the grading is $\eta_1 = -1$: At one loop it has one root $u = 0$ of each of the flavors 1 and 3. Due to its unpaired nature one would expect these two roots to be exactly $x = 0$ even at higher loops. This is however not what happens, instead they form a pair as $x_3 = -g^2/2x_1$ and are allowed to depart from $x = 0$ if $x_1 = \mathcal{O}(g) = x_3$. In fact, these particular roots turn out to have an expansion in odd powers of g unlike all the other roots.

5.4 Dynamic Bethe Ansatz

It is well-known that the number of fields in a local operator is not a conserved quantity at higher loops. Similarly, the $\mathfrak{u}(1)$ hypercharge B is anomalous and broken beyond one loop. In the spin chain picture this means that the Hamiltonian can add or remove sites of the spin chain. This dynamic type of spin chain appears to be an altogether novel model in the field of integrable spin chains. Despite some justified doubts – the interactions create and destroy particles and therefore appear not to be elastic – integrability seems to be an option even for dynamic spin chains [27]. A Bethe ansatz for this dynamic chain is expected to display some novel features. Here we will interpret the equations in Tab. 5 and explain why we believe that they originate from a dynamic chain.

The starting point is the wave function of an eigenstate of the Hamiltonian. For simplicity let us consider a dilute gas of excitations on a vacuum state. If there exists a Bethe ansatz of more or less familiar kind then the wave function must be completely described by a set of Bethe roots. Therefore one set of Bethe roots must be sufficient to describe states of different lengths or hypercharges. According to (5.7) a change in the length L or the hypercharge B, while keeping all the conserved Dynkin labels fixed, leads to modified excitation numbers K_j. This means that one set of Bethe roots *cannot*

describe mixed states with different L or B. This apparent dilemma can be solved by supplying a rule which maps between two different sets of Bethe roots when L or B are changed. Let us for definiteness increase both L and B by η_2

$$L \mapsto L + \eta_2, \qquad B \mapsto B + \eta_2. \tag{5.8}$$

Then only K_5 and K_7 change according to

$$K_5 \mapsto K_5 - 1, \qquad K_7 \mapsto K_7 + 1. \tag{5.9}$$

The simplest map from one set of Bethe roots to another changes one root of flavor 5 into a root of flavor 7. In this *dynamic transformation* the value of the root need not be invariant, here we propose the map[23]

$$x_7 = g^2/2x_5. \tag{5.10}$$

For cyclic spin chains not every set of Bethe roots is admissible, but the wave-function must be periodic. This is ensured by the Bethe equations: When one takes one excitation once around the trace, the net phase shift must be zero. The phases are obtained from permuting the excitation past all other excitations and all the sites of the vacuum. We have to make sure that the change of L or B is compatible with the periodicity conditions, i.e. Tab. 5. The roots x_5, x_7 appear in the Bethe equations for the roots of flavor 4 through 7. First of all, the Bethe equation for x_6 refers to x_5, x_7 only via u_5, u_7. The transformation (5.10), when mapped to the u-plane, reads

$$u_5 = u_7. \tag{5.11}$$

As u_5 and u_7 appear in precisely the same term, the Bethe equation for x_6 is left invariant. Next, the original Bethe equation for x_4 contains the term

$$\frac{x_4^{-\eta_2} - x_5}{x_4^{+\eta_2} - x_5} = \frac{x_4^{-\eta_2} - g^2/2x_7}{x_4^{+\eta_2} - g^2/2x_7} = \frac{x_4^{-\eta_2}}{x_4^{+\eta_2}} \, \frac{1 - g^2/2x_4^{+\eta_2}x_7}{1 - g^2/2x_4^{-\eta_2}x_7} = \left(\frac{x_4^-}{x_4^+}\right)^{\eta_2} \frac{1 - g^2/2x_4^{+\eta_2}x_7}{1 - g^2/2x_4^{-\eta_2}x_7}\,. \tag{5.12}$$

As we can see, this term is equivalent to the term for x_7 when the length L is increased by η_2. Finally, the roots of flavor 5 and 7 are fermionic and there are no interactions among different roots of these kinds. It is therefore left to confirm that the Bethe equation for x_5 itself is equivalent to the equation for x_7. The coupling to x_6 is via $u_5 = u_7$ and coincides for both flavors. The coupling to x_4 is

$$\prod_{j=1}^{K_4} \frac{x_5 - x_{4,j}^{+\eta_2}}{x_5 - x_{4,j}^{-\eta_2}} = \left(\prod_{j=1}^{K_4} \frac{x_{4,j}^+}{x_{4,j}^-}\right)^{\eta_2} \prod_{j=1}^{K_4} \frac{1 - g^2/2x_7 x_{4,j}^{+\eta_2}}{1 - g^2/2x_7 x_{4,j}^{-\eta_2}} = \prod_{j=1}^{K_4} \frac{1 - g^2/2x_7 x_{4,j}^{+\eta_2}}{1 - g^2/2x_7 x_{4,j}^{-\eta_2}}\,. \tag{5.13}$$

Again we see that the Bethe equation for x_7 is automatically satisfied. We however had to make use of the cyclicity constraint. Therefore the transformation (5.10) preserves

[23] The transformation requires one of the roots x_5, x_7 to be of $\mathcal{O}(g^2)$ if the other one is finite at $g = 0$. This is reminiscent of the discussion in [32], however, using the auxiliary roots x_5, x_7 instead of the main roots x_4. It would be interesting to see if there can be main roots $x_4 = \mathcal{O}(g^2)$ and what effect they have for the Bethe ansatz.

the periodicity of the state and the Bethe equations are consistent with changes of L and B. Of course, the dynamic transformation between x_5 and x_7 can be reversed

$$K_5 \mapsto K_5 \mp 1, \qquad K_7 \mapsto K_7 \pm 1, \qquad L \mapsto L \pm \eta_2, \qquad B \mapsto B \pm \eta_2. \tag{5.14}$$

Likewise we can transform on the other side of the Dynkin diagram

$$K_3 \mapsto K_3 \mp 1, \qquad K_1 \mapsto K_1 \pm 1, \qquad L \mapsto L \pm \eta_1, \qquad B \mapsto B \mp \eta_1 \tag{5.15}$$

using the same map as (5.10) between x_1 and x_3. This proves point *iv* by relying on the cyclic nature of the spin chain, i.e. in agreement with point *vii*.

We can even see from the Bethe equations that the length-changing effects are a genuinely higher-loop effect: The map $x \mapsto g^2/2x$ is singular in the limit $g \to 0$. A perfectly meaningful finite root x is mapped to the point $x = 0$ losing all the information of its origin. Thus there is only one configuration of the Bethe roots which can survive in the limit $g \to 0$: It is the one where all roots remain finite and do not approach $x = 0$.[24] The only case where the transformation remains meaningful at $g = 0$ is when $x_5 = 0$, $x_7 = \infty$ and there are no roots of flavor 6. It is related to multiplet shortening and has already been explained in Sec. 4.5, as $K_6 = 0$ restricts to that particular subsector. Here we understand why a root $x_5 = 0$ actually corresponds to a descendant: It is equivalent to $x_7 = \infty$ which represents a descendant of the state without x_7. For $g \neq 0$, the joining transformation of Sec. 4.5 turns into a regular dynamic transformation which confirms point *vi*.

Finally, let us consider point *ix*: What is special about $\mathfrak{psu}(2,2|4)$? The equations in Tab. 5 are a neat but very fragile arrangement of couplings of the sort "$x - x$" and "$1 - g^2/2xx$". The dynamic transformation acts between roots of two flavors. Their couplings to other flavors should be alike in order for the Bethe equations to be preserved by the transformation. In particular, they should couple to the momentum-carrying flavor in order to be related to the momentum constraint. This means that starting from the momentum-carrying node, the Dynkin diagram can extend only over three further consecutive nodes: the first node involved in the dynamic transformation, some other node, and the second node of the dynamic transformation. This last node couples back to the momentum-carrying one, but the link breaks at $g = 0$. When considering the very limited set of Dynkin diagrams for superalgebras, we see that we can have no more than two such triples of nodes originating from the momentum-carrying node. This is the case of $\mathfrak{psu}(2,2|4)$ as investigated in this article. Alternatively, one could have only one triple and potentially something else, i.e. $\mathfrak{su}(2|\mathcal{N})$ with $\mathcal{N} \geq 3$. For $\mathcal{N} = 3$ this would merely be a restriction of a $\mathfrak{psu}(2,2|4)$ model, see [27]. For $\mathcal{N} = 4$ we have the $\mathfrak{su}(2|4)$ plane-wave matrix model [19, 56, 42], which has the virtue of a real representation. Such an algebra with some non-compact signature could also play a role for integrable subsectors of less supersymmetric field theories at higher loops (if they exist). It would be interesting to see whether dynamic models for $\mathfrak{su}(2|\mathcal{N})$ with $\mathcal{N} > 4$, for an orthosymplectic algebra or for exceptional superalgebras exist.

[24] An exception is a pair of roots $x_{5,7} \sim g$ which cannot be transformed to something finite. In this case one of the roots will end up as $x_5 = 0$ and one as $x_7 = 0$.

5.5 Duality Transformation

To confirm point x of Sec. 5.1 we will now show that the Bethe equations for $\eta_2 = +1$ are equivalent to the ones for $\eta_2 = -1$. Consider the Bethe equation for roots $x_{5,k}$ of type 5

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k} - u_{6,j} + \frac{i}{2}}{u_{5,k} - u_{6,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5,k} - x^+_{4,j}}{x_{5,k} - x^-_{4,j}} . \tag{5.16}$$

It coincides with the equations for roots $x_{7,k}$ of type 7 when $x_{5,k}$ is replaced by $g^2/2x_{5,k}$

$$1 = \prod_{j=1}^{K_6} \frac{u_{7,k} - u_{6,j} + \frac{i}{2}}{u_{7,k} - u_{6,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{1 - g^2/2x_{7,k}x^+_{4,j}}{1 - g^2/2x_{7,k}x^-_{4,j}} \prod_{j=1}^{K_4} \frac{x^+_{4,j}}{x^-_{4,j}} \tag{5.17}$$

when the cyclicity constraint is imposed. Using the identities (2.33) we can transform the equation to the x-plane

$$\prod_{j=1}^{K_4} \frac{x_{5,k} - x^-_{4,j}}{x_{5,k} - x^+_{4,j}} \prod_{j=1}^{K_6} \frac{x_{5,k} - x^+_{6,j}}{x_{5,k} - x^-_{6,j}} \prod_{j=1}^{K_6} \frac{x_{5,k} - g^2/2x^+_{6,j}}{x_{5,k} - g^2/2x^+_{6,j}} = 1. \tag{5.18}$$

Let us consider a state with K_5 roots $x_{5,k}$ and K_7 roots $x_{7,k}$. Using the polynomial

$$\begin{aligned} P(x) = &\prod_{j=1}^{K_4} (x - x^+_{4,j}) \prod_{j=1}^{K_6} (x - x^-_{6,j}) \prod_{j=1}^{K_6} (x - g^2/2x^-_{6,j}) \\ &- \prod_{j=1}^{K_4} (x - x^-_{4,j}) \prod_{j=1}^{K_6} (x - x^+_{6,j}) \prod_{j=1}^{K_6} (x - g^2/2x^+_{6,j}) \end{aligned} \tag{5.19}$$

we can write the Bethe equation (5.18) as

$$P(x_{5,k}) = 0, \qquad P(g^2/2x_{7,k}) = 0. \tag{5.20}$$

Apart from these roots, there are further solutions: For counting purposes, we can consider $x = \infty$ to be a solution, it solves (5.18). It is associated to the cancellation of the terms $x^{K_4+2K_6}$ in (5.19). Furthermore $x = 0$ is a solution if the momentum constraint is satisfied. This can be viewed as a root of type 7 at $x = \infty$. The remaining solutions can be grouped into two classes. There are K_6 roots of the polynomial (5.19) which are of $\mathcal{O}(g^2)$ for small g. These are naturally associated to Bethe roots of type 7. The remaining $K_4 + K_6$ roots are of $\mathcal{O}(1)$ and thus correspond to Bethe roots of type 5. Therefore there are

$$\tilde{K}_5 = K_4 + K_6 - K_5 - 1, \qquad \tilde{K}_7 = K_6 - K_7 - 1 \tag{5.21}$$

further solutions which we denote by $\tilde{x}_{5,k}$ and $g^2/2\tilde{x}_{7,k}$, respectively. We now write the polynomial in factorized form as

$$P(x) \sim x \prod_{j=1}^{K_5} (x - x_{5,j}) \prod_{j=1}^{K_7} (x - g^2/2x_{7,j}) \prod_{j=1}^{\tilde{K}_5} (x - \tilde{x}_{5,j}) \prod_{j=1}^{\tilde{K}_7} (x - g^2/2\tilde{x}_{7,j}) \tag{5.22}$$

As before in Sec. 4.7 we use the two equivalent forms (5.19) and (5.22) of $P(x)$ to derive equations which translate between the two dual forms of the Bethe equations. The two relevant combinations of P are

$$\frac{P(x^+_{4,k})}{P(x^-_{4,k})}, \qquad \frac{P(x^-_{6,k})}{P(x^+_{6,k})}\,\frac{P(g^2/2x^-_{6,k})}{P(g^2/2x^+_{6,k})}. \tag{5.23}$$

They lead to

$$\begin{aligned}&\prod_{\substack{j=1\\j\neq k}}^{K_4}\left(x^+_{4,k}-x^-_{4,j}\right)\prod_{j=1}^{K_5}\frac{x^-_{4,k}-x_{5,j}}{x^+_{4,k}-x_{5,j}}\prod_{j=1}^{K_7}\frac{1-g^2/2x^-_{4,k}x_{7,j}}{1-g^2/2x^+_{4,k}x_{7,j}}\\ &=\prod_{\substack{j=1\\j\neq k}}^{K_4}\left(x^-_{4,k}-x^+_{4,j}\right)\prod_{j=1}^{\tilde K_5}\frac{x^+_{4,k}-\tilde x_{5,j}}{x^-_{4,k}-\tilde x_{5,j}}\prod_{j=1}^{\tilde K_7}\frac{1-g^2/2x^+_{4,k}\tilde x_{7,j}}{1-g^2/2x^-_{4,k}\tilde x_{7,j}}\end{aligned} \tag{5.24}$$

and

$$\begin{aligned}&\prod_{\substack{j=1\\j\neq k}}^{K_6}\frac{u_{6,k}-u_{6,j}-i}{u_{6,k}-u_{6,j}+i}\prod_{j=1}^{K_5}\frac{u_{6,k}-u_{5,j}+\frac{i}{2}}{u_{6,k}-u_{5,j}-\frac{i}{2}}\prod_{j=1}^{K_7}\frac{u_{6,k}-u_{7,j}+\frac{i}{2}}{u_{6,k}-u_{7,j}-\frac{i}{2}}\\ &=\prod_{\substack{j=1\\j\neq k}}^{K_6}\frac{u_{6,k}-u_{6,j}+i}{u_{6,k}-u_{6,j}-i}\prod_{j=1}^{\tilde K_5}\frac{u_{6,k}-\tilde u_{5,j}-\frac{i}{2}}{u_{6,k}-\tilde u_{5,j}+\frac{i}{2}}\prod_{j=1}^{\tilde K_7}\frac{u_{6,k}-\tilde u_{7,j}-\frac{i}{2}}{u_{6,k}-\tilde u_{7,j}+\frac{i}{2}}\end{aligned} \tag{5.25}$$

which are precisely the equations to turn the Bethe equations for roots of types 4 and 6 for $\eta_2=+1$ into the dual equations for $\eta_2=-1$ and dual roots $\tilde x_{5,j},\tilde x_{7,j}$. A similar argument applies to the duality between $\eta_1=+1$ and $\eta_1=-1$.

Note that these resulting equations are very restrictive for the structure of the Bethe equations:

- If we had assumed there to be no direct coupling between roots of 4 and 7 (or alternatively if we had started out with no roots of type 7), the dualization would have generated some new roots $\tilde x_{7,j}$ which couple to roots of type 4. It would not be clear how to interpret these additional roots then.
- The dualization (5.25) contains two self-scattering terms for roots of type 6. Contrary to the dualization used in [14] which has only one self-scattering term, this can only work if the roots of type 6 are bosonic. In fact, in the framework of [14], we dualize the (fermionic) roots of type 5 and 7 at the same time and thus generate two self-scattering terms for roots of type 6.[25]

Therefore, once we assume (5.16) as the Bethe equation for root of type 5 and we insist on the possibility of dualization, we can say that much of the structure of the remaining

[25] The dualization does not keep us from adding higher-loop self-scattering terms to the Bethe equation for roots of flavor 6. Whether it is reasonable to do so is a different question.

Bethe equations follows. It is reasonable to assume (5.16) because it agrees with the string sigma model in [12] in the thermodynamic limit.

For the time being, the dualization restricts us to the Dynkin diagrams in Fig. 1, but it would be interesting to see if the other Dynkin diagrams of $\mathfrak{psu}(2,2|4)$ can be realized and related by a more general duality transformation.

5.6 Frolov-Tseytlin Limit

In the thermodynamic limit the Bethe equations turn into integral equations. We find that the limit of the equations in Tab. 5 agrees with the generic form (see App. A for a dictionary)

$$\sum_{j=1}^{7} M_{k,j} \not{H}_j(x) + F_k(x) = -2\pi n_{k,a} \qquad \text{for } x \in \mathcal{C}_{k,a} \tag{5.26}$$

where $F_k(x)$ are some functions which specify the details of the model. The momentum constraint and the anomalous dimension are given by

$$G_4(0) = 2\pi n, \qquad \delta D = g^2 G_4'(0). \tag{5.27}$$

The potentials F_2, F_6 for the auxiliary bosonic nodes turn out to vanish

$$F_2(x) = F_6(x) = 0. \tag{5.28}$$

The remaining auxiliary potentials are all proportional to $G_4(g^2/2x) - G_4(0)$

$$-\eta_1 F_1(x) = +\eta_1 F_3(x) = +\eta_2 F_5(x) = -\eta_2 F_7(x) = G_4(g^2/2x) - G_4(0). \tag{5.29}$$

The most important function is the potential F_4 for the main excitations for which we find

$$\begin{aligned} F_4(x) = &-\eta_1 \left(G_1(g^2/2x) - \frac{G_1'(0)\, g^2/2x}{1 - g^2/2x^2} - \frac{G_1(0)}{1 - g^2/2x^2} \right) \\ &+ \eta_1 \left(G_3(g^2/2x) - \frac{G_3'(0)\, g^2/2x}{1 - g^2/2x^2} - \frac{G_3(0)}{1 - g^2/2x^2} \right) \\ &+ \eta_2 \left(G_5(g^2/2x) - \frac{G_5'(0)\, g^2/2x}{1 - g^2/2x^2} - \frac{G_5(0)}{1 - g^2/2x^2} \right) \\ &- \eta_2 \left(G_7(g^2/2x) - \frac{G_7'(0)\, g^2/2x}{1 - g^2/2x^2} - \frac{G_7(0)}{1 - g^2/2x^2} \right) + F_{\mathrm{g,s}}(x). \end{aligned} \tag{5.30}$$

It is the only potential which depends on the particular model, which is specified by the function σ, through the function $F_{\mathrm{g,s}}(x)$. For gauge theory the missing piece is the same as (2.57)

$$\begin{aligned} F_{\mathrm{g}}(x) = &\frac{L/x}{1 - g^2/2x^2} \\ &+ \tfrac{1}{2}(2 - \eta_1 - \eta_2) \left(2G_4(g^2/2x) - \frac{G_4'(0)\, g^2/2x}{1 - g^2/2x^2} - \frac{G_4(0)(2 - g^2/2x^2)}{1 - g^2/2x^2} \right). \end{aligned} \tag{5.31}$$

For string theory we recover the function (2.58)

$$F_{\mathrm{s}}(x) = \frac{L/x}{1-g^2/2x^2} - (\eta_1+\eta_2)\big(G_4(g^2/2x) - G_4(0)\big) + \tfrac{1}{2}(2+\eta_1+\eta_2)\frac{G_4'(0)\, g^2/2x}{1-g^2/2x^2} - \tfrac{1}{2}(2-\eta_1-\eta_2)\frac{G_4(0)\, g^2/2x^2}{1-g^2/2x^2}\,. \tag{5.32}$$

We can now recast the equations in the form used in [12]. This form is useful because it is closer to the underlying spectral curve and does not depend on a choice of Dynkin diagram. The curve is specified by the 4+4 quasi-momenta $\tilde{p}_k(x)$ and $\hat{p}_k(x)$ corresponding to the $\mathfrak{su}(4)$ and $\mathfrak{su}(2,2)$ parts of the algebra. The integral equations (5.26) become

$$\begin{aligned}
\tilde{p}\!\!\!/_l(x) - \tilde{p}\!\!\!/_k(x) &= 2\pi \tilde{n}_{kl,a} \quad \text{for } x \in \tilde{\mathcal{C}}_{kl,a},\\
\hat{p}\!\!\!/_l(x) - \hat{p}\!\!\!/_k(x) &= 2\pi \hat{n}_{kl,a} \quad \text{for } x \in \hat{\mathcal{C}}_{kl,a},\\
\tilde{p}\!\!\!/_l(x) - \hat{p}\!\!\!/_k(x) &= 2\pi n^*_{kl,a} \quad \text{for } x = x^*_{kl,a},
\end{aligned} \tag{5.33}$$

where $\tilde{\mathcal{C}}_{kl,a}, \hat{\mathcal{C}}_{kl,a}$ are branch cuts associated to the bosonic subalgebras $\mathfrak{su}(4), \mathfrak{su}(2,2)$, respectively. The points $x^*_{kl,a}$ specify fermionic excitations which cannot condense into cuts due to the Pauli principle [12]. The quasi-momenta are parametrized as follows

$$\begin{aligned}
\tilde{p}_k(x) &= \sum_{l=1}^{4}\big(\tilde{H}_{kl}(x) + H^*_{kl}(x)\big) + \varepsilon_k \tilde{F}(x) + F^*(x),\\
\hat{p}_k(x) &= \sum_{l=1}^{4}\big(\hat{H}_{lk}(x) + H^*_{lk}(x)\big) + \varepsilon_k \hat{F}(x) + F^*(x),
\end{aligned} \tag{5.34}$$

where the functions G_{kl}, H_{kl} and F are some combinations of the functions G_k, H_k and F_k. The coefficients ε_k equal $(+1,+1,-1,-1)$ for $k=(1,2,3,4)$. By comparing the two different formulations of the integral equations we obtain

$$\begin{aligned}
\tilde{F}(x) &= \tfrac{1}{2}F_4(x) - \tfrac{1}{4}(2-\eta_1-\eta_2)\eta_1 F_3(x),\\
\hat{F}(x) &= \tfrac{1}{2}F_4(x) + \tfrac{1}{4}(2+\eta_1+\eta_2)\eta_1 F_3(x).
\end{aligned} \tag{5.35}$$

while the resolvents G_{kl}, H_{kl} are related to G_k, H_k in a canonical way (cf. [12] for details). The fermionic potential F^* does not appear in the Bethe equations and we cannot determine it here. Let us introduce a couple of useful combinations

$$\begin{aligned}
\tilde{G}_{\mathrm{sum}} &= \tfrac{1}{2}\sum_{k,l=1}^{4} \varepsilon_k\big(\tilde{G}_{kl} + G^*_{kl}\big)\\
&= +\tfrac{1}{2}\eta_1(G_1 - G_3) + \tfrac{1}{2}\eta_2(G_7 - G_5) + \tfrac{1}{4}(2+\eta_1+\eta_2)G_4,\\
\hat{G}_{\mathrm{sum}} &= \tfrac{1}{2}\sum_{k,l=1}^{4} \varepsilon_k\big(\hat{G}_{lk} + G^*_{lk}\big)\\
&= +\tfrac{1}{2}\eta_1(G_1 - G_3) + \tfrac{1}{2}\eta_2(G_7 - G_5) - \tfrac{1}{4}(2-\eta_1-\eta_2)G_4,\\
G^*_{\mathrm{sum}} &= \tfrac{1}{2}\sum_{k,l=1}^{4} G^*_{kl} = +\tfrac{1}{2}\eta_1(G_1 - G_3) - \tfrac{1}{2}\eta_2(G_7 - G_5),\\
G_{\mathrm{mom}} &= \tilde{G}_{\mathrm{sum}} - \hat{G}_{\mathrm{sum}} = G_4
\end{aligned} \tag{5.36}$$

and similarly for H. We can then write the potentials for the gauge theory spin chain as

$$\tilde{F}_{\mathrm{g}}(x) = \frac{L/2x}{1-g^2/2x^2} + \frac{\hat{G}'_{\mathrm{sum}}(0)\, g^2/2x}{1-g^2/2x^2} + \frac{\hat{G}_{\mathrm{sum}}(0)}{1-g^2/2x^2} - \hat{G}_{\mathrm{sum}}(g^2/2x), \tag{5.37}$$

$$\begin{aligned}\hat{F}_{\mathrm{g}}(x) = \frac{L/2x}{1-g^2/2x^2} + \frac{\hat{G}'_{\mathrm{sum}}(0)\, g^2/2x}{1-g^2/2x^2} + \frac{\hat{G}_{\mathrm{sum}}(0)}{1-g^2/2x^2} - \hat{G}_{\mathrm{sum}}(g^2/2x) \\ + G_{\mathrm{mom}}(g^2/2x) - G_{\mathrm{mom}}(0).\end{aligned} \tag{5.38}$$

Note that this result, together with (5.33,5.34), agrees nicely with the conjectured higher-loop form of the $\mathfrak{so}(6)$ Bethe equations in the thermodynamic limit in [57]. The corresponding expressions for the string chain differ slightly

$$\begin{aligned}\tilde{F}_{\mathrm{s}}(x) &= \frac{L/2x}{1-g^2/2x^2} + \frac{\tilde{G}'_{\mathrm{sum}}(0)\, g^2/2x}{1-g^2/2x^2} + \frac{\hat{G}_{\mathrm{sum}}(0)}{1-g^2/2x^2} - \tilde{G}_{\mathrm{sum}}(g^2/2x) + G_{\mathrm{mom}}(0),\\ \hat{F}_{\mathrm{s}}(x) &= \frac{L/2x}{1-g^2/2x^2} + \frac{\tilde{G}'_{\mathrm{sum}}(0)\, g^2/2x}{1-g^2/2x^2} + \frac{\hat{G}_{\mathrm{sum}}(0)}{1-g^2/2x^2} - \hat{G}_{\mathrm{sum}}(g^2/2x).\end{aligned} \tag{5.39}$$

The latter potentials agree with the potentials of the string sigma model [12] and confirm point *iii*. The expansion of potentials around $x = \infty$ contributes the charges of the vacuum and the anomalous dimension to the Dynkin labels of a state, see point *viii*. The potentials start out with the same terms for both models

$$\tilde{F}(x) = \frac{L}{2x} + \mathcal{O}(1/x^2), \qquad \hat{F}(x) = \frac{L+\delta D}{x} + \mathcal{O}(1/x^2). \tag{5.40}$$

Let us finally investigate the transformation properties under the map $x \mapsto g^2/2x$. For string theory the potentials transform according to

$$\begin{aligned}\tilde{F}_{\mathrm{s}}(g^2/2x) &= -\tilde{F}_{\mathrm{s}}(x) - \tilde{H}_{\mathrm{sum}}(x) + G_{\mathrm{mom}}(0),\\ \hat{F}_{\mathrm{s}}(g^2/2x) &= -\hat{F}_{\mathrm{s}}(x) - \hat{H}_{\mathrm{sum}}(x).\end{aligned} \tag{5.41}$$

For definiteness we shall take the missing fermionic potential F^* from the string sigma model [12]

$$F^*(x) = G^{*\prime}_{\mathrm{sum}}(0)\frac{g^2/2x}{1-g^2/2x^2} + \frac{G^*_{\mathrm{sum}}(0)}{1-g^2/2x^2} - G^*_{\mathrm{sum}}(g^2/2x). \tag{5.42}$$

It transforms according to

$$F^*_{\mathrm{s}}(g^2/2x) = -F^*_{\mathrm{s}}(x) - H^*_{\mathrm{sum}}(x). \tag{5.43}$$

This leads to the following symmetry relations of the quasi-momenta for the thermodynamic limit of the string chain

$$\begin{aligned}\tilde{p}_k(g^2/2x) &= -\tilde{p}_{k'}(x) + \varepsilon_k G_{\mathrm{mom}}(0),\\ \hat{p}_k(g^2/2x) &= -\hat{p}_{k'}(x).\end{aligned} \tag{5.44}$$

Here the index k is maps to the permutation k' by $(1,2,3,4) \mapsto (2,1,4,3)$. For gauge theory we obtain

$$\begin{aligned}\tilde{F}_{\mathrm{g}}(g^2/2x) &= -\tilde{F}_{\mathrm{g}}(x) - \tilde{H}_{\mathrm{sum}}(x) + H_{\mathrm{mom}}(x),\\ \hat{F}_{\mathrm{g}}(g^2/2x) &= -\hat{F}_{\mathrm{g}}(x) - \hat{H}_{\mathrm{sum}}(x) + H_{\mathrm{mom}}(x) - G_{\mathrm{mom}}(0)\end{aligned} \tag{5.45}$$

and the quasi-momenta transform according to

$$\begin{aligned}\tilde{p}_k(g^2/2x) &= -\tilde{p}_{k'}(x) + \varepsilon_k H_{\mathrm{mom}}(x),\\ \hat{p}_k(g^2/2x) &= -\hat{p}_{k'}(x) + \varepsilon_k H_{\mathrm{mom}}(x) - \varepsilon_k G_{\mathrm{mom}}(0).\end{aligned} \tag{5.46}$$

This inversion appears to be the only difference between the gauge and string chain in the thermodynamic limit. This is because the Bethe equations follow from the analyticity properties (which are the same for both models) and the symmetry.

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A Thermodynamic Limit of Terms

In this appendix we present a dictionary of the various terms of the discrete Bethe ansatz, scattering phases and the resolvents in the thermodynamic limit. In this limit, the spectral parameter x and the coupling g are both considered to be large and of the same order as the length of the chain

$$x = \mathcal{O}(L), \qquad g = \mathcal{O}(L). \tag{A.1}$$

A.1 Scattering Phases

Let us first state the thermodynamic limit of the charges

$$q_r(x_k) = \frac{i}{r-1}\left(\frac{1}{(x_k^+)^{r-1}} - \frac{1}{(x_k^-)^{r-1}}\right) = \frac{1}{1 - g^2/2x_k^2}\,\frac{1}{x_k^r} + \mathcal{O}(1/L^{r+1}). \tag{A.2}$$

Now we consider two Bethe roots x_k, x_j. Their interactions can be expressed in terms of scattering phases. Several useful combinations to write the phases are given by

$$\begin{aligned}
\psi(x_k, x_j) &= -i \log \frac{x_k - x_j^+}{x_k - x_j^-} = \sum_{r=1}^{\infty} x_k^{r-1} q_r(x_j), \\
\theta_{r,s}(x_k, x_j) &= (\tfrac{1}{2} g^2)^{(r+s-1)/2}\, q_r(x_k)\, q_s(x_j), \\
\theta(x_k, x_j) &= \sum_{r=2}^{\infty} (\theta_{r,r+1} - \theta_{r+1,r}), \\
\varPsi(x_k, x_j) &= -i \log \frac{u_k - u_j - \frac{i}{2}}{u_k - u_j + \frac{i}{2}}\,.
\end{aligned} \tag{A.3}$$

In the thermodynamic limit, all the scattering phases are $\mathcal{O}(1/L)$. The phases introduced above are approximated by

$$\begin{aligned}
\psi(x_k, x_j) &= \frac{1}{1 - g^2/2x_j^2} \frac{1}{x_j - x_k} + \mathcal{O}(1/L^2), \\
\theta_{r,s}(x_k, x_j) &= \frac{1}{1 - g^2/2x_k^2} \frac{1}{1 - g^2/2x_j^2} \frac{(\frac{1}{2} g^2)^{(r+s-1)/2}}{x_k^r x_j^s} + \mathcal{O}(1/L^2), \\
\theta(x_k, x_j) &= \frac{g^2/2x_k^2}{1 - g^2/2x_k^2} \frac{g^2/2x_j^2}{1 - g^2/2x_j^2} \frac{1}{1 - g^2/2x_k x_j} \frac{x_k - x_j}{x_k x_j} + \mathcal{O}(1/L^2), \\
\varPsi(x_k, x_j) &= \frac{1}{u_j - u_k} + \mathcal{O}(1/L^2) = \sum_{r=1}^{\infty} u_k^{r-1} u_j^{-r} + \mathcal{O}(1/L^2).
\end{aligned} \tag{A.4}$$

The main and auxiliary phases are related by

$$\varPsi = \psi + \theta + \theta_{1,2} + \mathcal{O}(1/L^2). \tag{A.5}$$

For the main scattering terms we obtain

$$-i \log \frac{x_k^a - x_j^c}{x_k^b - x_j^d} = \tfrac{1}{2}(b + c - a - d)\psi + \tfrac{1}{2}(b - a)\theta_{1,2} + \tfrac{1}{2}(b - a)\theta_{2,1} + \mathcal{O}(1/L^2), \tag{A.6}$$

where $a, b, c, d = 0, \pm 1$ distinguish between $x, x^\pm$. The limit of the auxiliary terms yields

$$-i \log \frac{1 - g^2/2x_k^a x_j^c}{1 - g^2/2x_k^b x_j^d} = \tfrac{1}{2}(b + c - a - d)\theta + \tfrac{1}{2}(c - d)\theta_{1,2} + \tfrac{1}{2}(a - b)\theta_{2,1} + \mathcal{O}(1/L^2). \tag{A.7}$$

The scattering terms in the u-plane limit to

$$-i \log \frac{u_k - u_j + \frac{i}{2} a}{u_k - u_j + \frac{i}{2} b} = \tfrac{1}{2}(b - a)\, \varPsi(x_k, x_j) + \mathcal{O}(1/L^2). \tag{A.8}$$

This is compatible with the above expressions using the identity (A.5). The combination σ for the string chain yields the auxiliary phase

$$-i \log \sigma(x_k, x_j) = \theta(x_k, x_j). \tag{A.9}$$

Finally, the limit of the potential term is

$$-i \log \left(\frac{x_k^a}{x_k^b}\right)^L = \frac{\frac{1}{2}(a-b)L/x_k}{1-g^2/2x_k^2} + \mathcal{O}(1/L). \tag{A.10}$$

A.2 Resolvents

Let us introduce one resolvent for the x-plane and one for the u-plane

$$G(x) = \sum_{k=1}^{K} \frac{1}{1-g^2/2x_k^2} \frac{1}{x_k - x}, \qquad H(x) = \sum_{k=1}^{K} \frac{1}{u_k - u(x)}. \tag{A.11}$$

The two resolvents are related by the identity

$$H(x) = G(x) + G(g^2/2x) - G(0). \tag{A.12}$$

We can also write them in an integral form using a density $dx\,\rho(x) = du\,\rho(u)$

$$G(x) = \int \frac{dy\,\rho(y)}{1-g^2/2y^2} \frac{1}{y-x}, \qquad H(x) = \int \frac{dv\,\rho(v)}{v-u(x)}. \tag{A.13}$$

The resolvents are related to the summed main scattering phases

$$\sum_{j=1}^{K} \psi(x_k, x_j) = G(x_k), \qquad \sum_{j=1}^{K} \Psi(x_k, x_j) = H(x_k). \tag{A.14}$$

The partial auxiliary scattering phases yield derivatives of the resolvent at $x = 0$

$$\sum_{j=1}^{K} \theta_{r,s}(x_k, x_j) = \frac{(\frac{1}{2}g^2)^{(r+s-1)/2}/x_k^r}{1-g^2/2x_k^2}\, G^{(s-1)}(0). \tag{A.15}$$

Finally, the total auxiliary phase translates to

$$\sum_{j=1}^{K} \theta(x_k, x_j) = G(g^2/2x_k) - G(0) - \frac{g^2/2x_k}{1-g^2/2x_k^2}\, G'(0), \tag{A.16}$$

so that (A.5,A.14,A.12) match up. This dictionary lets us compute the thermodynamic limit of all expressions straightforwardly.

B Transfer Matrices

B.1 Rank-One Sectors

Before we test our asymptotic extrapolation, let us introduce an important object for integrable models, a transfer matrix. Within Bethe ansätze there often exist expressions for the eigenvalues of the transfer matrices in terms of the Bethe roots. In fact, the

Bethe equations follow from these equations by demanding that the transfer matrix has no poles. For eigenstates with singular Bethe roots (usually at $x = 0, \pm\frac{i}{2}$) the cancellation of poles gives the correct prescription for regularizing the Bethe equations. For the three models we heuristically find for the eigenvalues of the fundamental transfer matrix

$$
\begin{aligned}
T_{\text{fund}}(x) = +\eta_1 \left(\frac{x^+}{x}\right)^L \prod_{j=1}^{K} \left(\frac{x^{-\eta_1} - x_j^{+\eta_1}}{x - x_j} \, \frac{1 - g^2/2x^- x_j^+}{1 - g^2/2xx_j} \, \sigma^{-1}(x, x_j) \right) \\
+\eta_2 \left(\frac{x^-}{x}\right)^L \prod_{j=1}^{K} \left(\frac{x^{+\eta_2} - x_j^{-\eta_2}}{x - x_j} \, \frac{1 - g^2/2x^+ x_j^-}{1 - g^2/2xx_j} \, \sigma^{+1}(x, x_j) \right). \qquad \text{(B.1)}
\end{aligned}
$$

Here we use a compact notation with two parameters $\eta_1, \eta_2 = \pm 1$. Together they determine the model with the total grading $\eta = (\eta_1 + \eta_2)/2$. For $\eta = \eta_1 = \eta_2 = +1$ it agrees with the expression given in [29]. The Bethe equations (2.39) follow from cancelling the poles at x_j. The overall factor is ambiguous, we have chosen it so that the terms appear in a symmetric way.[26]

Thermodynamic Limit. In the thermodynamic limit, the transfer matrix gives a sum of exponentials

$$
T_{\text{fund}}(x) = \eta_1 \exp\bigl(ip_1(x)\bigr) + \eta_2 \exp\bigl(ip_2(x)\bigr). \qquad \text{(B.2)}
$$

The exponents are called quasi-momenta, for gauge theory we obtain

$$
\begin{aligned}
p_1(x) &= +\eta_1 H(x) + \frac{L/2x}{1 - g^2/2x^2} + \tfrac{1}{2}(1 - \eta_1) F_{\text{g}}(x), \\
p_2(x) &= -\eta_2 H(x) - \frac{L/2x}{1 - g^2/2x^2} - \tfrac{1}{2}(1 - \eta_2) F_{\text{g}}(x) \qquad \text{(B.3)}
\end{aligned}
$$

with the potential

$$
F_{\text{g}}(x) = 2G(g^2/2x) - G(0) \, \frac{2 - g^2/2x^2}{1 - g^2/2x^2} - G'(0) \, \frac{g^2/2x}{1 - g^2/2x^2} \, . \qquad \text{(B.4)}
$$

For string theory, the quasi-momenta read

$$
\begin{aligned}
p_1(x) &= +\eta_1 G(x) + \frac{L/2x}{1 - g^2/2x^2} - \tfrac{1}{2}(1 - \eta_1) F_{\text{s1}}(x) + \tfrac{1}{2}(1 + \eta_1) F_{\text{s2}}(x), \\
p_2(x) &= -\eta_2 G(x) - \frac{L/2x}{1 - g^2/2x^2} + \tfrac{1}{2}(1 - \eta_2) F_{\text{s1}}(x) - \tfrac{1}{2}(1 + \eta_2) F_{\text{s2}}(x) \qquad \text{(B.5)}
\end{aligned}
$$

where the potentials are given by

$$
F_{\text{s1}}(x) = G(0) \, \frac{g^2/2x^2}{1 - g^2/2x^2}, \qquad F_{\text{s2}}(x) = G'(0) \, \frac{g^2/2x}{1 - g^2/2x^2} \, . \qquad \text{(B.6)}
$$

These quasi-momenta agree with the expressions for classical strings investigated derived in [9, 10, 12] (when fixing $B = 0$ in the fermionic case $\eta_1 \neq \eta_2$).

[26] One might be tempted to remove the denominators $1 - g^2/2xx_j$ in order to eliminate poles at $x = g^2/2x_j$. As we do not know how to derive these expressions from first principles, we cannot decide which form is more suitable.

B.2 The $\mathfrak{su}(1,1|2)$ Sector

The transfer matrix for the $\mathfrak{su}(1,1|2)$ spin chain appears to be

$$\begin{aligned}
T_{\text{fund}}(x) = &-\eta_1 \left(\frac{x^+}{x}\right)^L \prod_{j=1}^{K_1} \frac{x^{+\eta_1} - x_{1,j}}{x^{-\eta_1} - x_{1,j}} \prod_{j=1}^{K_2} \left(\frac{1 - g^2/2x^- x_{2,j}^+}{1 - g^2/2x^- x_{2,j}^-}\, \sigma^{-1}(x, x_{2,j}) \right) \\
&+\eta_1 \left(\frac{x^+}{x}\right)^L \prod_{j=1}^{K_1} \frac{x^{+\eta_1} - x_{1,j}}{x^{-\eta_1} - x_{1,j}} \prod_{j=1}^{K_2} \left(\frac{1 - g^2/2x^- x_{2,j}^+}{1 - g^2/2x x_{2,j}}\, \sigma^{-1}(x, x_{2,j})\, \frac{x^{-\eta_1} - x_{2,j}^{+\eta_1}}{x - x_{2,j}} \right) \\
&+\eta_2 \left(\frac{x^-}{x}\right)^L \prod_{j=1}^{K_3} \frac{x^{-\eta_2} - x_{3,j}}{x^{+\eta_2} - x_{3,j}} \prod_{j=1}^{K_2} \left(\frac{1 - g^2/2x^+ x_{2,j}^-}{1 - g^2/2x x_{2,j}}\, \sigma^{+1}(x, x_{2,j})\, \frac{x^{+\eta_2} - x_{2,j}^{-\eta_2}}{x - x_{2,j}} \right) \\
&-\eta_2 \left(\frac{x^-}{x}\right)^L \prod_{j=1}^{K_3} \frac{x^{-\eta_2} - x_{3,j}}{x^{+\eta_2} - x_{3,j}} \prod_{j=1}^{K_2} \left(\frac{1 - g^2/2x^+ x_{2,j}^-}{1 - g^2/2x^+ x_{2,j}^+}\, \sigma^{+1}(x, x_{2,j}) \right). \qquad \text{(B.7)}
\end{aligned}$$

The Bethe equations (4.35) follow from the above expression $T_{\text{fund}}(x)$ by cancelling the poles at $x = x_{1,k}^{+\eta_1}, x_{2,k}, x_{3,k}^{-\eta_2}$.

Dualization. Let us verify that the above expression for the transfer matrix is valid for all choices of η_1, η_2. Due to the relation (4.41) the following identity holds

$$P(x^{-\eta_1}) \prod_{j=1}^{K_1} \frac{x^{+\eta_1} - x_{1,j}}{x^{-\eta_1} - x_{1,j}} = P(x^{+\eta_1}) \prod_{j=1}^{\tilde{K}_1} \frac{x^{-\eta_1} - \tilde{x}_{1,j}}{x^{+\eta_1} - \tilde{x}_{1,j}} \qquad \text{(B.8)}$$

Using the original definition (4.40) of the polynomial, the first two lines of $T_{\text{fund}}(x)$ in (B.7) can be written as

$$P(x^{-\eta_1}) \prod_{j=1}^{K_1} \frac{x^{+\eta_1} - x_{1,j}}{x^{-\eta_1} - x_{1,j}} \left(\frac{x^+}{x}\right)^L \prod_{j=1}^{K_2} \left(\frac{1 - g^2/2x^- x_{2,j}^+}{1 - g^2/2x x_{2,j}}\, \sigma^{-1}(x, x_{2,j})\, \frac{1}{x - x_{2,j}} \right) \qquad \text{(B.9)}$$

When we flip the sign η_1 and use $\tilde{x}_{1,k}$ instead of $x_{1,k}$ the first two lines in $T_{\text{fund}}(x)$ become equivalent to

$$P(x^{+\eta_1}) \prod_{j=1}^{\tilde{K}_1} \frac{x^{-\eta_1} - \tilde{x}_{1,j}}{x^{+\eta_1} - \tilde{x}_{1,j}} \left(\frac{x^+}{x}\right)^L \prod_{j=1}^{K_2} \left(\frac{1 - g^2/2x^- x_{2,j}^+}{1 - g^2/2x x_{2,j}}\, \sigma^{-1}(x, x_{2,j})\, \frac{1}{x - x_{2,j}} \right) \qquad \text{(B.10)}$$

The same applies to dualization of x_3-roots. Therefore $T_{\text{fund}}(x)$ remains valid after the duality transformation.

Thermodynamic Limit. The thermodynamic limit of the transfer matrix for $\eta_1 = \eta_2 = +1$ can be written as

$$\begin{aligned}
p_1(x) &= -G_1(x) - F_1(x) \\
p_2(x) &= -G_1(x) + G_2(x) - F_1(x) \\
p_3(x) &= +G_3(x) - G_2(x) + F_3(x), \\
p_4(x) &= +G_3(x) + F_3(x). \qquad \text{(B.11)}
\end{aligned}$$

The potential for gauge theory is

$$F_{\mathrm{g},j} = -\frac{L/2x}{1-g^2/2x^2} - G_2(g^2/2x_k) + G_2(0) + G_j'(0)\,\frac{g^2/2x_k}{1-g^2/2x_k^2} + G_j(0)\,\frac{g^2/2x_k^2}{1-g^2/2x_k^2}\,, \quad \text{(B.12)}$$

whereas the one for the string chain reads

$$F_{\mathrm{s},j} = -\frac{L/2x}{1-g^2/2x^2} - G_2'(0)\,\frac{g^2/2x_k}{1-g^2/2x_k^2} + G_j(0)\,\frac{g^2/2x_k^2}{1-g^2/2x_k^2} + G_j'(0)\,\frac{g^2/2x_k}{1-g^2/2x_k^2}\,. \quad \text{(B.13)}$$

The latter expression agrees with classical superstrings on $AdS_3 \times S^3$ found in [12].

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